Probabilistic Selling in Vertically Differentiated Markets

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This paper studies two fundamental questions regarding probabilistic selling in vertically differentiated markets: When is it profitable and how does one design it optimally? For the first question, we identify an important but overlooked economic mechanism driving probabilistic selling in vertically differentiated markets: the convexity of consumer preferences. In stark contrast to the literature finding that probabilistic selling is *never* profitable except in the presence of certain capacity constraint or consumer bounded rationality, we find that with many alternative utility functions capable of representing convex preference, probabilistic selling is *always* profitable. For the second question, we study the optimal strategy of probabilistic selling, including the design and price of the probabilistic good and the prices of the component goods. We show that under some technical conditions, the optimal price of the high-quality component good increases while the optimal price of the low-quality component good decreases upon the introduction of probabilistic selling, thereby increasing the market coverage and the economic efficiency without launching an actual new product line. To further illustrate the design of probabilistic selling, we use an example based on the canonical utility function, which is widely used in the economics literature on vertical product differentiation. We derive a closed-form solution to the problem of optimal probabilistic selling. We also take advantage of the analytical tractability of the canonical utility to further explore the design of multiple probabilistic goods.

Key words: probabilistic selling; vertically differentiated market; preference convexity

1. Introduction

A probabilistic good is a synthetic good consisting of a mix, often in the form of a lottery, between existing goods (hereafter, component goods). The selling of probabilistic goods, a practice known as probabilistic selling or opaque selling, is an innovative way of combining products or services that are mostly homogeneous but differ in one important attribute, either horizontally or vertically.

Earlier works (Jiang 2007, Fay and Xie 2008, 2010, Jerath et al. 2010) on probabilistic selling focus on horizontally differentiated component goods such as sweaters of different colors or flights at different times of the day between the same pair of origin and destination cities. These studies help explain the economic forces driving probabilistic selling in numerous horizontal markets. As noted in another pioneering work on probabilistic selling (Zhang et al. 2015), “there are equally numerous quality-differentiated markets where consumers strictly prefer one product over the other” and “it is important to ask whether probabilistic selling will prove profitable in quality-differentiated markets.” In fact, the practice of selling a synthetic product has been documented in the internet broadband service industry, where high- and low-speed services are mixed as a probabilistic good

(Zhang et al. 2015, Zheng et al. 2019); in the car rental industry, where cars of different sizes are mixed (Zheng et al. 2019); in the airline industry, where seats of different classes[1] are mixed (Zhang et al. 2015); in the hotel industry, where hotels[2] of different star ratings are mixed; in the online retailing industry, where earphones of different qualities are mixed (Zheng et al. 2019). Given the practices of probabilistic selling in such markets, Zhang et al. (2015) argue probabilistic selling is a profitable way for the disposal of the excess capacity of the high-quality goods. Alternatively, Huang and Yu (2014) and Zheng et al. (2019) suggest bounded rationality, whether because of anecdotal reasoning or overweighting of salient attributes by consumers' salient thinking, may also lead to the emergence of probabilistic selling in vertically differentiated markets. The current paper contributes to this new stream of work on probabilistic selling.

The first main contribution of this paper is the revelation of an important but overlooked economic factor underlying the profitability of probabilistic selling in vertically differentiated markets: *convexity* of consumer preferences. Unlike Huang and Yu (2014) and Zheng et al. (2019), consumers are fully rational in our model. Different from Zhang et al. (2015) who assume that there is excess capacity for the high-quality component goods but insufficient capacity for the low-quality component goods, we do not impose a constraint on the capacity of either component good. The extant literature would suggest that probabilistic selling cannot be profitable in our setting. However, we show that once we move beyond linear utility function and consider preference convexity, probabilistic selling can be profitable for many utility functions. We further identify a sufficient condition that guarantees the profitability of probabilistic selling, based on our proposed concept of λ-concavity which is a joint property of two distinct objects: consumer preference and consumer type distribution.

To understand the basic intuition underlying the importance of preference convexity, consider a risk-neutral seller who evaluates the profitability of offering a probabilistic good by mixing two component goods of different qualities before any price or capacity optimization. Because of the opportunity cost, the seller is unwilling to sell the probabilistic good at a price *below* the average price of the component goods weighted by the mixing probability. So, other than possibly risk lovers, why would any rational consumer, instead of consuming one of the component goods, ever be interested in consuming the probabilistic good at a price *above* the weighted average price? The reason, we believe,

[1] For example, airlines often sell "upgradable" tickets which allows the ticket holder, with some positive probability, to upgrade seat class. Such an "upgradability" can be implemented in different ways. For example, consumers may pay a fare higher than the cost of a "non-upgradable" ticket for this privilege. For most airlines, basic economy tickets, which are sold at the lowest price, are non-upgradable tickets. Customers may also join an elite membership program for such a privilege, and membership can be obtained upon achieving enough reward points or through direct purchase. In general, an upgrade is not guaranteed and the probability of upgrade may also differ among different consumers. For instance, American Airlines (AA) offers AAdvantage elite program, which consists of four different membership statuses. A higher status level increases the chance of being upgraded.

[2] Such a program is often referred to as a *roulette*. A participant of the program pays a discounted price for a stay either in a (relatively) high-end or a (relatively) low-end hotel that is not revealed to the participant before the booking. See, for example, Delphina's Prestige Roulette Formula (`https://www.delphinahotels.co.uk/sardinia/roulette.html`) or H10's Tenerife Roulette (`https://www.h10hotels.com/en/tenerife-hotels/roulette-tenerife`).

is that the consumer dislikes extreme allocation of her budget between quality improvement of the focal good and the consumption of other goods. This important property of consumer preference, formalized in economics as the crucial concept of preference convexity, is not an uncommon technical property. Rather, it is a central premise capturing the fundamental principle of diminishing marginal rates of substitution.

The extant literature all focus on the linear utility function which cannot represent a (strictly) convex preference. We believe this largely contributes to the negative findings[3] in the literature. For example, Zhang et al. (2015) find probabilistic selling is *never* optimal without excess capacity of the high-quality component goods, whereas Huang and Yu (2014) and Zheng et al. (2019) find probabilistic selling is *never* optimal unless they introduce some form of bounded rationality. The linear utility function, popular in the literature, and was originally suggested as a linear approximation to more realistic utility functions for analytical tractability. However, once we stop using linear approximation, these negative findings start to disappear. For example, if consumers have Cobb-Douglas utility functions and their types are uniformly distributed, probabilistic selling is *always* profitable. Hence, this paper complements the extant literature theoretically by revealing the crucial role of preference convexity in probabilistic selling in vertically differentiated markets.

Two practical implications of this theoretical finding are immediate. First, because preference convexity is well accepted in economics and is commonly assumed for consumer theory in standard microeconomics textbooks (Varian 1992, Mas-colell et al. 1995, Jehle and Reny 2011, Kreps 1992, 2013, Pindyck and Rubinfeld 2018), our finding suggests probabilistic selling can be more widely utilized by firms that offer quality-differentiated products. Hence, probabilistic selling can potentially be profitable in contexts currently overlooked in academia and by practitioners. Second, we believe the proposed economic mechanism complements the current explanations of probabilistic selling in vertically differentiated markets, especially the one based on the excess (insufficient) capacity of the high-quality (low-quality) component goods.[4] Indeed, the existence of such an asymmetric mismatch between the supply of and the demand for quality-differentiated component goods is unclear. Excess capacity, especially of high-quality component goods, is costly, and a profit-maximizing firm should have a strong incentive to match its capacity with the demand, especially in the long run. Moreover, Zheng et al. (2019) show that once we model consumer types as continuous instead of dichotomous as in Zhang et al. (2015), probabilistic selling is never optimal even with excess (insufficient) capacity of the high-quality (low-quality) component goods.

[3] See Proposition 1 in Huang and Yu (2014), Lemma 1 in Zhang et al. (2015), and Proposition 1 in Zheng et al. (2019).

[4] The size of the variable transaction cost specifically linked to probabilistic good also plays an important role in the explanation based on excess capacity. For example, Zhang et al. (2015) find that only with a sufficiently large transaction cost can the offering of both component goods along with the probabilistic good become optimal. However, in reality, we do observe the offering of probabilistic goods and both component goods even when such a variable transaction cost is negligible, at least compared with the prices of component goods.

To aid the application of probabilistic selling in vertically differentiated markets, we develop a theory of optimal probabilistic selling, which is the second main contribution of the paper. First, we study the implications of probabilistic selling on the optimal prices of the component goods without using any specific functional form of the utility. We prove under some regularity conditions that the optimal price of the high-quality component good increases while the optimal price of the low-quality component good decreases, as the result of optimal probabilistic selling. Therefore, the practice of probabilistic selling can increase the market coverage and the economic efficiency without launching an actual new product line. This insight is useful both to practitioners and future researchers whose choice of utility function will inevitably vary depending on market characteristics. Second, we illustrate the optimal design of probabilistic selling using the canonical utility function that is often adopted in the economics literature to study vertical market differentiation. The explicit and simple form of the utility function allows us to derive a closed-form solution to the problem of optimal probabilistic selling. Third, we explore the optimal design of probabilistic selling when the seller can offer multiple levels of probabilistic goods, which the extant literature has called for but not yet studied.

The rest of the paper is organized as follows. In Section 2, we briefly review previous studies on probabilistic selling in order to position the current paper in the literature. In Section 3, we set up the model with a generic two-attribute utility function to reveal the importance of preference convexity and the sufficiency of λ-concavity for probabilistic selling to be profitable without introducing bounded rationality. In Section 4, we develop a theory of optimal probabilistic selling and illustrate the optimal design in details with a fully-solved example. Finally, we conclude the paper in Section 5 with a discussion of its contributions, managerial implications, and limitations.

2. Literature Review

This literature can be broadly categorized based on two important modeling choices: whether the component goods are horizontally differentiated or vertically differentiated, and whether the model assumes rational consumers or consumers with bounded rationality. Accordingly, we list in Table 1 some representative works in each category and review those most closely related to the current paper. For a more comprehensive review of the literature on probabilistic selling, we refer interested readers to Jerath et al. (2009) or Zhang et al. (2015).

Naturally, most of the academic literature on probabilistic selling concerns two fundamental questions regarding the phenomenon: Under what conditions can probabilistic selling be profitable, and how does one optimally design probabilistic goods? Jiang (2007) and Fay and Xie (2008) are among the earliest works on probabilistic selling, and they focus on the case in which the two component goods are horizontally differentiated. For example, Jiang (2007) considers a Hotelling model with two component goods (e.g., morning flights and afternoon flights for the same origin-destination pair)

	Horizontally Differentiated	**Vertically Differentiated**
Rational	Jiang (2007), Fay and Xie (2008, 2010), Jerath et al. (2010, 2009), Shapiro and Shi (2008)	Zhang et al. (2015)(with excess capacity), current paper (without excess capacity)
Bounded Rationality	Huang and Yu (2014)	Huang and Yu (2014), Zheng et al. (2019)

Table 1 Literature position of the current paper

placed at the two ends of the Hotelling line. The paper shows that probabilistic selling, with equal probability of selecting the two component goods, can sometimes improve profit by essentially price discriminating against consumers with less flexibility in terms of the choice between the two component goods. Fay and Xie (2008) also adopt the Hotelling framework. They find that probabilistic selling strictly improves profit if the marginal cost of the (symmetric) component goods is sufficiently low. They also find the optimal mixing probability of the component goods to be exactly 0.5, thereby providing a justification for the equal-probability assumption in Jiang (2007). In addition, Fay and Xie (2008) also reveal that demand uncertainty and mismatch between capacity and demand can also motivate the use of probabilistic selling. Our paper differs from these papers mainly in our focus on vertically differentiated component goods rather than horizontally differentiated component goods.

More recently, researchers started to investigate the emergence of probabilistic selling in vertically differentiated markets. Huang and Yu (2014) first show probabilistic selling is never optimal when homogeneous consumers have rational expectations. However, if consumers have bounded rationality as is captured by anecdotal reasoning, probabilistic selling can be optimal. Similarly, Zheng et al. (2019) study probabilistic selling by taking into account consumers' salient thinking behavior. They show probabilistic selling is never profitable with rational consumers but does improve the seller's profit with salient thinkers. Different from these behavioral economics models, Zhang et al. (2015) study probabilistic selling in vertically differentiated markets with *rational* consumers, which is also assumed in the current paper. They find probabilistic selling can be profitable only if the capacity of the high-quality (low-quality) component good exceeds (falls below) the market demand. In contrast to findings from these studies, our paper suggests probabilistic selling can be profitable in more general situations with *neither* bounded rationality *nor* asymmetric capacity constraint. Such a different finding is rooted in the fact that in the extant literature (Huang and Yu 2014, Zhang et al. 2015, Zheng et al. 2019), consumer preference is modeled by a utility function that does not satisfy the property of strict convexity, which is an important factor for probabilistic selling to be profitable in vertically differentiated markets.

In a sense, our paper bridges the gap between the literature on probabilistic selling in horizontally differentiated markets and the literature on probabilistic selling in vertically differentiated markets. The extant literature poses a puzzle regarding the applicability of probabilistic selling in these two market settings. For horizontally differentiated markets, probabilistic selling seems to be well justified in a wide variety of markets, but for vertically differentiated markets, probabilistic selling seems to only work in special circumstances (e.g., bounded rationality, salient thinking, excess (insufficient) capacity of high-quality (low-quality) component goods). An important insight from our paper is that such an asymmetry between horizontally differentiated markets and vertically differentiated markets is largely artificial, driven by the restrictive model of consumer preference selected in the extant literature to study vertically differentiated markets.

3. Profitability of Probabilistic Selling

We consider a focal good that is an indivisible good with its quality denoted by q. Each consumer has a unit demand for the focal good. Before introducing a probabilistic good, the focal good has two quality levels, q_H and q_L, offered at prices p_H and p_L, respectively. Without loss of generality, we assume $q_H > q_L > q_0$ and $p_H > p_L$, where a quality level of $q_0 > 0$ simply means no consumption of the focal good. The unit cost of the high- and low-quality goods are c_H and c_L, respectively.

We model consumer preference using a generic two-attribute utility function, $U(x,y)$, that represents a consumer's preference over the consumption of Hicks' composite good (x), whose price is normalized to 1, and the focal good (y). For the Hicks' composite good, the consumer chooses the quantity (x) to consume, whereas for the focal good, the consumer chooses to either consume at one of the two quality levels or not to consume at all. For example, for a consumer planning on a vacation, y may refer to the hotel quality (e.g., star rating, room type) or flight quality (e.g., economy class, business class), and x may refer to the remaining budget for all other goods and services during the vacation.

Consumers are heterogeneous in their budget level w, which has a distribution that is absolutely continuous with its support normalized to $[0,1]$. Let $F:[0,1]\to[0,1]$ be the cumulative distribution function (CDF) of w. The consumer's choice problem is the following, where $p(y)$, the price of the focal good, is either p_H or p_L if the consumer purchases, or 0 otherwise:

$$\max_{x\geq 0, y\in\{q_0,q_H,q_L\}} U(x,y), \quad \text{s.t.} \quad x+p(y)\leq w.$$

Clearly, the consumer chooses to spend the budget to maximize her utility, resulting in the utility of

- $U(w-p_H, q_H)$ if the consumer purchases the high-quality focal good,
- $U(w-p_L, q_L)$ if the consumer purchases the low-quality focal good, or
- $U(w, q_0)$ if the consumer does not purchase the focal good.

In the absence of a probabilistic good, the consumer maximizes her utility by comparing the above three utility levels.

We assume the utility function is strictly increasing in each attribute, and impose the following regularity conditions on its structure:

1. $\partial^2 U/\partial x \partial y > 0$
2. $\partial^2 U/\partial x^2 \leq 0$
3. $\partial^2 U/\partial y^2 \leq 0$.

The first regularity condition, known as the single-crossing condition or the Spence Mirrlees property (Milgrom and Shannon 1994), is often assumed in the literature[5]. Depending on its interpretation as $\partial^2 U/\partial x \partial y$ or $\partial^2 U/\partial y \partial x$ which are mathematically equivalent for a C^2 function, there are two perspectives to its intuition. We may think of it as a modeling technique to guarantee that two utility curves as functions of consumer budget or consumer type (i.e., $w \mapsto U(w - p(y), y)$) corresponding to the consumption of two goods of different quality levels only cross each other once, thereby naturally generating the kind of market segmentation often observed in business. This is graphically illustrated in Figure 1 where any pair of the three curves (i.e., $w \mapsto U(w, q_0)$, $w \mapsto U(w - p_H, q_H)$, $w \mapsto U(w - p_H, q_H)$) only crosses each other once. Alternatively, we may think of this regularity condition as a stylized way to capture the observation that consumers who consume more in general (because of a larger budget) are typically more willing to pay for quality improvement (i.e., with a larger $\partial U/\partial y$). The second and third conditions capture the basic economic principle of diminishing marginal utility.

Let w_0 be the budget level at which a consumer is indifferent[6] between the high- and low-quality component goods, i.e. $U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L)$. Clearly, w_0 is an implicit function of p_H and p_L. An immediate consequence[7] of the regularity conditions is that consumers with $w < w_0$ strictly prefer the low-quality good to the high-quality good, and consumers with $w > w_0$ strictly prefer the high-quality good to the low-quality good. Similarly, let $\underline{w}$ be the budget level at which a consumer is indifferent between buying the low-quality component good and not buying at all, i.e., $U(\underline{w}, q_0) = U(\underline{w} - p_L, q_L)$. Before the introduction of probabilistic selling, the market is partitioned into three segments, which is illustrated in the example of Figure 1. In this example, because $q_H > q_L > q_0$, by Assumption 1, the slope of the curve $w \mapsto U(w - p_L, q_L)$ is larger than the slope of the curve $w \mapsto U(w, q_0)$. Similarly, the slope of the curve $w \mapsto U(w - p_H, q_H)$ is larger than the slope of the curve $w \mapsto U(w - p_L, q_L)$. With the budget interval large enough, the three curves intersect, generating three segments: $[0, \underline{w}]$, $(\underline{w}, w_0]$, and $(w_0, 1]$. Consumers with $w \in [0, \underline{w}]$ choose not to purchase the

[5] For example, Shaked and Sutton (1987) assume the condition to study vertical product differentiation and industrial structure. Similarly, Zhang et al. (2015) assume it, albeit in a discrete setting. More specifically, in their context of two types of consumers, this regularity condition becomes $V_{HH} - V_{HL} > V_{LH} - V_{LL}$, where V_{ij} is the value of product type j to consumers of type i, where $i, j \in \{H, L\}$.

[6] The existence of w_0 is guaranteed as long as the budget interval is sufficiently large, which we assume throughout the paper. Otherwise, consumers either all prefer the high-quality good or all prefer the low-quality good, thereby excluding the very existence of a vertically differentiated market in the first place. Similar conditions have been imposed in the literature of vertical differentiation (Tirole 1988, Mussa and Rosen 1978) to ensure positive demand for products of different qualities.

[7] To see this, note $U(w - p_H, q_H) - U(w - p_L, q_L) = \big(U(w - p_H, q_H) - U(w - p_H, q_L)\big) - \big(U(w - p_L, q_L) - U(w - p_H, q_L)\big)$ is monotone increasing in w because the term in the first parentheses is increasing due to the first regularity condition, whereas the term in the second parentheses is decreasing due to the second regularity condition.

focal good (i.e., consuming quality level q_0); consumers with $w \in (\underline{w}, w_0]$ choose to purchase the low-quality focal good (i.e., consuming quality level q_L) ; and consumers with $w \in (w_0, 1]$ choose to purchase the high-quality focal good (i.e., consuming quality level q_H).

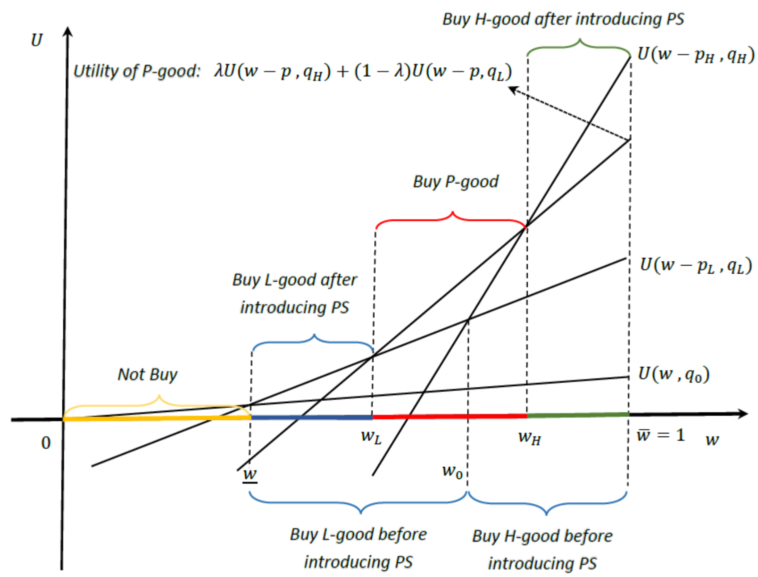

Figure 1 **An illustration of market segmentation based on consumer heterogeneity in their budget level w.**

We represent a probabilistic good by the pair (p, λ) where p is the price of the probabilistic good and $\lambda \in (0, 1)$ is the mixing probability—the probability of receiving the high-quality good. The expected utility from purchasing this probabilistic good for a consumer with budget level w is $\lambda U(w - p, q_H) + (1 - \lambda)U(w - p, q_L)$. Let w_H (w_L) be the budget level[8] at which a consumer is indifferent between the probabilistic good and the high-quality (low-quality) component good, respectively, i.e.,

$$U(w_H - p_H, q_H) = \lambda U(w_H - p, q_H) + (1 - \lambda)U(w_H - p, q_L) \tag{1}$$

$$U(w_L - p_L, q_L) = \lambda U(w_L - p, q_H) + (1 - \lambda)U(w_L - p, q_L). \tag{2}$$

In Figure 1, the expected utility from purchasing the probabilistic good is depicted as the curve $w \mapsto \lambda U(w - p, q_H) + (1 - \lambda)U(w - p, q_L)$ which intersects with the curve $w \mapsto U(w - p_L, q_L)$ at w_L and with the curve $w \mapsto U(w - p_H, q_H)$ at w_H.

With the introduction of a probabilistic good, consumers essentially have an intermediate quality level to choose and some may find it the best choice if the price of the probabilistic good is not too high. This is the case in the illustration of Figure 1 where the curve $w \mapsto \lambda U(w - p, q_H) + (1 -$

[8] Similar to the logic for the existence of w_0, the existence of w_H and w_L is guaranteed as long as the support of the budget distribution is large enough which we assume throughout the paper.

$\lambda)U(w-p,q_L)$ is depicted as above the other curves in the region $[w_L,w_H]$. In other words, consumers with budget levels $w\in[w_L,w_H]$ prefer the probabilistic good to the component goods. This region $[w_L,w_H]$ represent the market segment cannibalized by the introduction of probabilistic good, which consists of some consumers who would have chosen the high-quality component good in the absence of the probabilistic good (i.e., those with $w\in[w_0,w_H]$), as well as some consumers who would have chosen the low-quality component good in the absence of the probabilistic good (i.e., those with $w\in[w_L,w_0]$). Therfore, the question of whether the demand for a probabilistic good is positive translates graphically to the question of whether the curve $w\mapsto\lambda U(w-p,q_H)+(1-\lambda)U(w-p,q_L)$ is above the other curves in some region. The following proposition characterizes the condition under which the demand for the probabilistic good (p,λ) is positive.

Proposition 1 (Pivotal Consumer) *The demand for probabilistic good (p,λ) is positive if and only if $p<\bar{p}(\lambda)$ where $\bar{p}(\lambda)$ uniquely solves the following equation of p:*

$$\lambda U(w_0-p,q_H)+(1-\lambda)U(w_0-p,q_L)=U(w_0-p_H,q_H)=U(w_0-p_L,q_L) \tag{3}$$

Equivalently, the demand for the probabilistic good (p,λ) is positive if and only if a consumer with budget level w_0 strictly prefers the probabilistic good (p,λ) to either component good in which case $w_L<w_0<w_H$. Moreover, $\bar{p}(\lambda)>\lambda p_H+(1-\lambda)p_L$.

The above result suggests that whether the demand for the probabilistic good is positive crucially depends on how those consumers with budget level w_0 (henceforth the *pivotal consumers*) rank the probabilistic good.

3.1. The Importance of Preference Convexity

In this subsection, we demonstrate the importance of preference convexity for probabilistic selling to be profitable and offer our explanation of the development of the current literature. Besides this theoretical insight, an important practical implication is that the potential application of probabilistic selling is beyond what the current literature has identified. First, recall the mathematical definition of preference convexity.

Definition 1 *A preference relation $\succeq$ on $\mathcal{X}$ is convex if for every $x\in\mathcal{X}$, the upper contour set $\{y\in\mathcal{X}:y\succeq x\}$ is convex; that is, if $y\succeq x$ and $z\succeq x$, then $\lambda y+(1-\lambda)z\succeq x$ for any $\lambda\in[0,1]$. The preference relation $\succeq$ on $\mathcal{X}$ is strictly convex if for every x, $y\succeq x$, $z\succeq x$, and $y\neq z$, we have $\lambda y+(1-\lambda)z\succ x$ for all $\lambda\in(0,1)$.*

Zhang et al. (2015) explains the profitability of probabilistic selling in vertically differentiated markets without introducing bounded rationality. The key factor there is the excess capacity of the high-quality component good and the insufficient capacity of the low-quality component goods. Such a result is obtained by assuming two types of consumers. However, Zheng et al. (2019) pointed out that with a continuous distribution of consumer types, probabilistic selling is never profitable for any level of capacity constraint. This negative result indirectly demonstrates the importance of modeling consumer preference as strictly convex because the result is obtained by assuming a linear preference, and classical economic theory typically only considers rational preference that is either strictly convex or weakly convex (i.e., linear). This indirect approach is analytically tractable thanks to the simplicity of linear utility function.

One drawback of this indirect approach is that the intuition for the role of preference convexity in probabilistic selling is obscured, which might have explained why the literature immediately examined the possible role of bounded rationality in probabilistic selling. Our goal in this subsection is to directly demonstrate the importance of preference convexity, using a generic utility function to model consumer preference. Unlike the indirect approach, explicitly solving for and comparing the solutions of two optimization problem involving a utility function that is not linear, hence abstract, is difficult, if possible at all. To circumvent this difficulty, we instead consider a benchmark setting where the seller offers the probabilistic good without expanding capacity and adjusting the prices of component goods. Such a setting reflects a short-term scenario where the capacity is relatively fixed and the price of the component goods are relatively stable. For practitioners, this is likely the first scenario to consider while evaluating the profitability of probabilistic selling. Although highly stylized, this scenario can clearly reveal the importance of preference convexity, as is illustrated in the next proposition.

Proposition 2 *Without capacity expansion and price adjustment for component goods, probabilistic selling is profitable only if consumer preference is strictly convex.*

Intuitively, offering probabilistic goods gives consumers the option of an intermediate quality level at an intermediate price. Such an option can only be attractive to consumers if they are averse to "extreme" allocation of their budget between the focal good and the composite good, which is the essence of strict convexity of preference.

To better illustrate the crucial role of preference convexity in the emergence of probabilistic goods, we explain the underlying logic graphically in Figure 2 where the vertical axis represents the quality of the focal good and the horizontal axis represents the remaining budget spent on the composite good. Because consumer utility function is strictly increasing in both attributes, moving northeast to an indifference curve will increase utility. The consumption portfolios from purchasing the high- and low-quality goods correspond to point A and B, respectively. It's easy to see that point H in the figure, representing the portfolio $(w_0 - \underline{p}, \lambda q_H + (1-\lambda) q_L)$ where $\underline{p} \equiv \lambda p_H + (1-\lambda) p_L$ is the price lower bound from the proof of Proposition 2, must lie on the line segment AB. The probabilistic

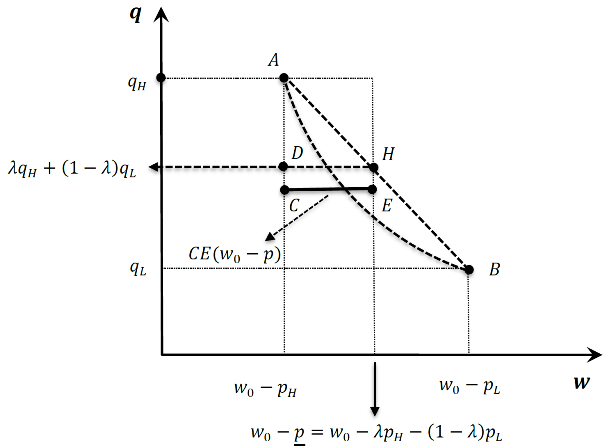

Figure 2 A graphical proof of Proposition 2 based on indifference curve analysis of the pivotal consumers.

good will only be offered at a price greater than $\underline{p}$. Hence, the expectation of the probabilistic good must lie on the line segment DH. Because $\partial^2 U(x,y)/\partial y^2 \leq 0$, the certainty equivalent (CE) of the probabilistic good must lie below DH. This certainty equivalent is represented by the segment[9] CE in the figure. By Proposition 1, the probabilistic good has positive demand if and only if it is strictly preferred by the pivotal consumer. Because the pivotal consumer's indifference curve connects A and B, for her to strictly prefer a point on the segment CE to A (or B), the indifference curve must bulge toward the origin. This roughly translates to strictly convex preference, because a preference is strictly convex if the upper contour set $\{y \in \mathcal{X} : y \succeq x\}$ is strictly convex.

Considered "a formal expression of basic inclination of economic agent for diversification" (Mas-colell et al. 1995), preference convexity is a central concept in economics, implying that the marginal rate of substitution (MRS) diminishes along the indifference curve because we can expect that a consumer will prefer to give up fewer and fewer units of a second good to get additional units of the first one. A rational preference is (strictly) convex if and only if it can be represented by a (strictly) quasiconcave[10] utility function, which is related to but more general than (strictly) concave utility functions. Strict convexity of consumer preference simply means the preference is convex and not degenerate (i.e., linear). With convex preference, an agent prefers holding a mix of two extreme portfolios to holding either of them. Therefore, our analysis suggests the profitability of probabilistic selling is related to rational consumers' desire for diversification: they don't want to hold an extreme portfolio of either definitely/always consuming low-quality goods or definitely/always paying a high price.

[9] CE is plotted as a *parallel line* segment, which is true for the Cobb-Douglas utility function. However, in general, CE does not even need to be linear. For Cobb-Douglas Utility $U(x,y) = x^a y^b$, where $a, b < 1$, we have $\lambda(w_0 - p)^a q_H^b + (1-\lambda)(w_0 - p)^a q_L^b = (w_0 - p)^a CE^b$, which implies $CE = (\lambda q_H^b + (1-\lambda) q_L^b)^{1/b}$, which is a constant less than $\lambda q_H + (1-\lambda) q_L$.

[10] A strictly quasiconcave utility is often defined as a function $f : \mathbb{R}^n \to \mathbb{R}$, such that $\forall x \neq y$ and $\lambda \in (0,1)$ with $f(x) \geq f(y)$, $f\big(\lambda x + (1-\lambda) y\big) > f(y)$.

The notion of preference convexity is well accepted in economics, and microeconomics textbooks (Varian 1992, Mas-colell et al. 1995, Jehle and Reny 2011, Kreps 1992, 2013, Pindyck and Rubinfeld 2018) commonly assume it for the development of consumer theory. For example, Varian (1992) argues (p. 157) that "if demand functions are well defined and everywhere continuous and are derived from preference maximization, then the underlying preference must be strictly convex." In other words, in microeconomics theory, a well-behaved demand function requires that consumers have a strict convex preference over a bundle of goods. Hence, preference convexity is also important in the theory of general equilibrium. As Varian (1992) (p. 393) argues, "Usually, the assumption of strict convexity has been used to assure that the demand function is well-defined – that there is only a single bundle demanded at each price – and that the demand function is continuous – that small changes in prices give rise to small changes in demand. The convexity assumption appears to be necessary for the existence of an equilibrium allocation since it is easy to construct examples where nonconvexities cause discontinuities of demand and thus nonexistence of equilibrium prices."

The extant literature of probabilistic selling on vertically differentiated markets began by modeling consumer preference using the utility function $\theta q - p$, where θ denotes consumer type, in order to show probabilistic selling is never optimal unless one introduces behavioral factors in consumer's choice model (Huang and Yu 2014, Zheng et al. 2019) or certain capacity constraints (Zhang et al. 2015). Unfortunately, this particular utility function cannot represent strictly convex preference, because its indifference curve is linear. Popular in the literature largely due to its tractability, this utility function is motivated in Tirole (1988) (p. 97) as an approximation of a more general utility function in order to facilitate the study of product differentiation and pricing in vertically differentiated markets. To see the nature of this approximation, note the utility function $\theta q - p$ is equivalent[11] to $q - \theta p$ if θ is interpreted as consumer type. Now consider a more general utility function $U(w-p, q) = u(w-p) + q$, where $u(\cdot)$ is continuously differentiable. By the mean value theorem, $U(w-p, q) = u(w) - pu'(\hat{w}) + q$, where $w - p < \hat{w} < w$. Because $u(w)$ plays no role in the consumer's choice problem, the above utility is equivalent to $q - u'(\hat{w})p$ where $u'(\hat{w})$ is interpreted as *consumer type* θ. Importantly, note $\theta \equiv u'(\hat{w})$ is actually a function of not just w, but also the price p. By modeling the utility function directly as $q - \theta p$ as if θ were only determined by the budget w, hence representing consumer type, one ignores the dependency of θ on p which is often referred to as the income effect. The tractability from ignoring the income effect often outweighs the resulting loss of generality. But for probabilistic selling, this has a profound impact. The very nature of *linear* approximation of the consumer utility function *guarantees* the violation of strict preference convexity, which, according to Proposition 2, is an important factor driving the profitability of probabilistic selling in vertically differentiated

[11] Indeed, we can simply redefine it by its reciprocal and note an affine transformation of a utility function does not change the underlying preference.

markets. Although the simplicity of the utility function $q-\theta p$ is likely innocuous in most applications, for probabilistic selling on vertically differentiated markets, it turns out to be consequential.

3.2. The Sufficiency of λ-Concavity

Now that we understand the importance of preference convexity to the profitability of probabilistic selling, it is natural to identify some sufficient condition for the profitability of probabilistic selling, which will complement the insight of the previous subsection. From now on, we allow the seller to freely adjust its capacity and price of each component goods so as to maximize its profit in a long-term equilibrium. To characterize the sufficient condition, we first define the concept of λ-concavity which describes a pair $(\succeq, F)$ where $\succeq$ is a rational preference and F is a probability distribution.

Definition 2 *Fix (p_H, p_L, q_H, q_L). Let $\succeq$ be a two-attribute preference represented by the utility function $U(x,y)$ that satisfies the regularity conditions, and F be a cumulative distribution function. The pair $(\succeq, F)$ is called λ-concave for $\lambda \in (0,1)$, if*

$$F(y) > \lambda F(x) + (1-\lambda)F(z). \tag{4}$$

where $x < y < z$ are unique solutions to the following equations:

$$\begin{aligned}
U(x-p_L, q_L) &= \lambda U\big(x-\lambda p_H-(1-\lambda)p_L, q_H\big)+(1-\lambda)U\big(x-\lambda p_H-(1-\lambda)p_L, q_L\big)\\
U(y-p_H, q_H) &= U(y-p_L, q_L)\\
U(z-p_H, q_H) &= \lambda U\big(z-\lambda p_H-(1-\lambda)p_L, q_H\big)+(1-\lambda)U\big(z-\lambda p_H-(1-\lambda)p_L, q_L\big).
\end{aligned}$$

For ease of exposition, we also say a preference (distribution) is λ-concave with a distribution (preference) if the pair is λ-concave. In our context, the triple (x,y,z) corresponds to (w_L, w_0, w_H) for the probabilistic selling strategy $(\lambda p_H + (1-\lambda)p_L, \lambda)$.

Let (p_h^*, p_l^*) be the optimal prices of the component good without probabilistic selling. The following result states a sufficient condition for probabilistic selling to be profitable.

Proposition 3 *Probabilistic selling is profitable if the consumer preference and their budget distribution are λ-concave for some $\lambda \in (0,1)$ at (p_h^*, p_l^*), in which case the probabilistic selling strategy $(p_h^*, p_l^*, \lambda p_h^* + (1-\lambda)p_l^*, \lambda)$ generates more profit than the optimal strategy without probabilistic selling.*

To understand the intuition without getting into the technical details, we summarize the proof of Proposition 3 here. First, we can show that at the optimal prices (without probabilistic selling) p_h^* and p_l^*, selling the probabilistic good $(\lambda p_h^* + (1-\lambda)p_l^*, \lambda)$ yields higher profit margin in the segment (w_L, w_0) but lower profit margin in the segment (w_0, w_H), compared with not selling the probabilistic good. If the structure of consumer preference is λ-concave with the distribution of w, the ratio between the number of consumers in the segment (w_L, w_0) and the number of consumers in the segment (w_0, w_H) is sufficiently high so that the overall profit gain from probabilistic selling is positive. To appreciate the implication of Proposition 3, we consider two examples.

EXAMPLE 1. Consider the linear utility function $\theta q - p$ and interpret F as the distribution of θ. It's straightforward to verify that

$$y = \frac{p_H - p_L}{q_H - q_L}, \quad x = \frac{\lambda p_H + (1-\lambda)p_L - p_L}{q_H - q_L} \cdot \frac{1}{\lambda} = y, \quad z = \frac{p_H - \lambda p_H - (1-\lambda)p_L}{q_H - q_L} \cdot \frac{1}{1-\lambda} = y$$

Hence, $x = y = z$, which implies the linear utility function is not λ-concave with any distribution.

This example partially explains the negative finding in the extant literature that probabilistic selling is generally not profitable if consumers have rational preferences. Our next example offers a stark contrast to this literature finding by showing the general profitability of probabilistic selling for the well-known family of Cobb-Douglas utility[12] functions, which is frequently used by standard microeconomics textbooks[13] to illustrate the classical demand theory.

EXAMPLE 2. The family of Cobb-Douglas utility functions and the uniform distribution are 1/2-concave for any (p_H, p_L, q_H, q_L). Hence, the probabilistic selling strategy $(p_h^*, p_l^*, (p_h^* + p_l^*)/2, 1/2)$ generates more profit than the optimal strategy without probabilistic selling, i.e., (p_h^*, p_l^*).

The drastic difference between Example 1 and 2 illustrate the importance of modeling preference beyond linearity when we study probabilistic selling in vertically differentiated markets. We conclude this section with a positive finding that further demonstrates this point.

Corollary 1 *Given any $\lambda \in (0,1)$ and any strictly quasiconcave utility function satisfying the regularity conditions, there exists a distribution such that it is λ-concave with the preference. Such a distribution can always be chosen as absolutely continuous.*

The proof is straightforward once we rewrite (4) as

$$\frac{F(y) - F(x)}{F(z) - F(y)} > \frac{1-\lambda}{\lambda}.$$

Indeed, given $\lambda \in (0,1)$, the unique solutions x, y, z are completely determined by the preference and not affected by the distribution. Because $x < y < z$, we can always construct a distribution such that the ratio between the probability mass in $[x, y]$ and the probability mass in $[y, z]$ is greater than $(1-\lambda)/\lambda$.

[12] A Cobb-Douglas utility function has the form of $U(x,y) = x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}}$ with $\alpha, \beta \geq 1$. All Cobb-Douglas utility functions are strictly quasiconcave. Indeed, recall quasiconcavity is preserved by an increasing transformation and any (strictly) concave function is (strictly) quasiconcave. The mapping $x \mapsto e^x$ is monotone increasing, and any function of the form $a \ln x + b \ln y$ with $a, b > 0$ is strictly concave because its Hessian matrix is clearly negative definite. By redefining $q^{\frac{1}{\beta}}$ as q in our problem, we can assume $\beta = 1$ without loss of generality. Hence, we parameterize a Cobb-Douglas utility function as $U(x,y) = x^{\frac{1}{\alpha}} y$ in the paper.

[13] See, for example, Varian (1992), Mas-colell et al. (1995), Jehle and Reny (2011), Kreps (1992, 2013), Pindyck and Rubinfeld (2018).

4. Design and Market Implication of Probabilistic Selling

In this section, we address our second research question: the optimal design and the market implication of probabilistic selling. Our objective in this section is twofold. First, because the choice of consumer utility function depends on market contexts, and for many realistic utility functions, no closed-form solution for the optimal probabilistic good (p, λ) exists, characterizing the implications of probabilistic selling on the optimal prices of the component goods without actually solving for the optimal (p, λ) will be valuable to future researchers and practitioners. Our second objective is to illustrate the optimal probabilistic selling through an example for which we have a closed-form solution. Denote $c \equiv c_H - c_L$ for ease of notation throughout this section.

4.1. General Model

The problem of optimal probabilistic selling can be framed as a two-stage decision problem where the seller chooses the prices of the component goods during the first stage, and design the probabilistic good (p, λ) in the second stage. We solve the optimal decision problem through backward induction.

Because we characterize the optimal probabilistic selling without assuming specific utility function, in order to have some tractability, we follow the literature (Gabszewicz and Thisse 1979, Shaked and Sutton 1982, 1983) by assuming uniform distribution of w. The optimal design of the probabilistic good, given the prices of the two component goods, is the following optimization problem where the objective function is the profit gain from the introduction of the probabilistic good:

$$\max_{p \leq \bar{p},\ \lambda \in (0,1)} \pi(p, \lambda) = \begin{array}{l} \big(p - (\lambda c_H + (1-\lambda) c_L)\big)(w_H - w_L) \\ -(p_H - c_H)(w_H - w_0) - (p_L - c_L)(w_0 - w_L). \end{array} \tag{5}$$

Recall w_0 is an implicit function of p_H and p_L defined by $U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L)$. Given a pair (p_H, p_L), denote the optimal design of the probabilistic good by (p^*, λ^*) and the corresponding optimal profit gain by $\pi^*(p_H, p_L)$ to emphasize that it's a function of p_H and p_L. We first characterize the comparative statics of the optimal profit gain with respect to the prices of the component goods in the following result.

Proposition 4 (Comparative Statics) *For any price pair of the component goods* (p_H, p_L),

$$\frac{\partial \pi^*(p_H, p_L)}{\partial p_H} + \frac{\partial \pi^*(p_H, p_L)}{\partial p_L} = 0.$$

The above result suggests the optimal profit gain as a function of component-good prices is always increasing in one while decreasing in the other and at the *same rate*, regardless of the prices of the component goods. Note Proposition 4 is a local property and does not imply the monotonicity of $\pi^*(p_H, p_L)$, either in p_H or in p_L.

Denote by π^0 the profit without probabilistic selling, i.e., $\pi^0(p_H,p_L)=(p_H-c_H)\big(1-w_0\big)+(p_L-c_L)\big(w_0-\underline{w}\big)$. The first-stage problem is the following pricing problem where the objective, $\Pi(p_H,p_L)$, defined as the sum of the profit without probabilistic selling and the profit gain from probabilistic selling, represents the total profit given the price pair (p_H,p_L):

$$\max_{p_H,p_L} \Pi(p_H,p_L) \equiv \pi^0(p_H,p_L)+\pi^*(p_H,p_L). \tag{6}$$

Recall (p_h^*,p_l^*) denotes the optimal price pair of the component goods without probabilistic selling, i.e., $(p_h^*,p_l^*)=\arg\max_{p_H,p_L}\pi^0(p_H,p_L)$. Let (p_H^*,p_L^*) be the optimal prices of the component goods with probabilistic selling, i.e., $(p_H^*,p_L^*)=\arg\max_{p_H,p_L}\Pi(p_H,p_L)$.

Of particular interest is the comparison between (p_H^*,p_L^*) and (p_h^*,p_l^*). If the seller adopts probabilistic selling to increase profit and do so optimally, will such a strategy improve economic efficiency? To shed light on this important question, we need to understand whether the market coverage increases as the result of probabilistic selling. More specifically, we need to compare p_L^* with p_l^* to see whether the price of the low-quality component good would decrease. If $p_L^*<p_l^*$, some consumers who previously could not afford the focal good can now afford it, hence are better off. Those consumers who would purchase the low-quality component good in the absence of probabilistic selling are also better off with the introduction of probabilistic selling. Indeed, if they continue to purchase the low-quality component good, they are better off because they now purchase the same product at a lower price. If they instead purchase the probabilistic good, they gain even more because they prefer the probabilistic good to the low-quality component good, the purchase of which can already give them higher utility levels than before.

To compare (p_H^*,p_L^*) with (p_h^*,p_l^*) without actually solving them, we need the objective function to be well-behaved globally. Otherwise, the optimal prices may not exist, and may not be unique even if they exist. The simplest technical requirement is for the objective function to be strictly concave so that we can use the first-order conditions to characterize (p_H^*,p_L^*). Our next result states a sufficient condition for probabilistic selling to improve market efficiency.

Proposition 5 (Efficiency) *If $\underline{w}$ is linear in p_L and the following two conditions are satisfied,*

$$\frac{\partial w_H}{\partial p_H}\Big(2-\frac{\partial p^*}{\partial p_H}+c\frac{\partial \lambda^*}{\partial p_H}\Big)>\frac{1}{2}+\big(p^*-p_H+(1-\lambda^*)c\big)\frac{\partial^*}{\partial p_H}\Big(\frac{\partial w_H}{\partial p_H}\Big), \quad \forall (p_H,p_L), \tag{7}$$

$$(p-p_L-\lambda c)\frac{\partial w_L}{\partial p_L}>w_L-\underline{w}-(p_L-c_L)\frac{\partial \underline{w}}{\partial p_L} \qquad \text{at } (p_H,p_L)=(p_h^*,p_l^*), \tag{8}$$

the optimal price of the high-quality (low-quality) component good increases (decreases) upon the introduction of probabilistic selling, with $p_H^*-p_h^*\geq p_l^*-p_L^*$.

Condition (7) is a sufficient condition for the profit function $\Pi(p_H, p_L)$ to be strictly concave. Note we introduced the notation $\partial^*/\partial p_H$ to indicate that our evaluation of the partial derivative needs to consider p^* and/or λ^* as implicit functions of p_H. On the other hand, the usual notation $\partial/\partial p_H$ is what one usually expects for the evaluation of partial derivative. The notational distinction becomes necessary for our analysis because we use the Envelope Theorem in the proof of Proposition 5.

Condition (8) is the necessary and sufficient condition for the optimal profit gain from probabilistic selling to increase (decrease) in p_H (p_L) at $(p_H, p_L) = (p_h^*, p_l^*)$ which is the optimal pair of component-good prices without probabilistic selling. In practice, we may verify condition (8) by first numerically solving (p^*, λ^*) in the second-stage problem, given the current (optimal) prices of the component goods, and then evaluating $\partial w_L/\partial p_L$ and w_L at $(p_h^*, p_l^*, p^*, \lambda^*)$.

Given that the objective function is strictly concave, and thanks to some of its structural properties, the optimal prices of the component goods with probabilistic selling can be obtained by increasing the price of the high-quality component good and decreasing the price of the low-quality component good. The linearity assumption of $\underline{w}$ as a function of p_L is required for the application of a technical result we developed to prove the proposition. For example, the assumption is satisfied if consumers have a Cobb-Douglas utility function $U(x,y) = x^{\frac{1}{\alpha}} y$ in which case $\underline{w}(p_L) = p_L \cdot q_L^\alpha/(q_L^\alpha - q_0^\alpha)$.

4.2. Example: Canonical Utility Function

In this section, we illustrate the design of probabilistic selling using a particular utility function as an example. This utility function, $U(x,y) = xy$, which we refer to as the *canonical utility function*, was introduced in Gabszewicz and Thisse (1979). This canonical utility function is often used in the economics literature to study vertical product differentiation (Gabszewicz and Thisse 1979, Shaked and Sutton 1982, 1983, Sutton 1986, Gabszewicz et al. 1986, Shaked and Sutton 1987, Bolton and Bonanno 1988, Fraja 1996). It is instrumental in the development of the theory of natural oligopolies and natural monopoly in economics (Waterson 1987).

The canonical utility function is both rich enough to represent strictly convex preference, and simple enough to allow for analytical tractability. The closed-form solution with this utility function not only allows us to illustrate the optimal design of probabilistic selling, but also enables us to explore the design of an arbitrary number of probabilistic goods, which is the first direction of future research emphasized by Zhang et al. (2015) in their conclusion.

For the canonical utility function, we can solve for the following values based on their definitions in Section 3, where $\gamma \equiv q_H/(q_H - q_L) > 1$:

$$w_0 = \gamma p_H + (1-\gamma) p_L, \quad w_H = \frac{p_H}{1-\lambda}\gamma - \frac{p}{1-\lambda}(\gamma - 1 + \lambda), \quad w_L = \frac{p}{\lambda}(\gamma - 1 + \lambda) + \frac{p_L}{\lambda}(1-\gamma).$$

We can also explicitly solve for $\bar{p}(\lambda)$ as the following:

$$\bar{p}(\lambda) = \frac{\lambda\gamma}{\gamma - 1 + \lambda} p_H + \Big(1 - \frac{\lambda\gamma}{\gamma - 1 + \lambda}\Big) p_L$$

We first design the optimal probabilistic good given the prices of the two component goods. We then use backward induction to explicitly solve for the optimal prices of the two component goods. With the canonical utility function, we have $w_0 = \lambda w_L + (1-\lambda) w_H$ which allows us to write the optimization problem of (5) as the following:

$$\begin{aligned} \max_{p,\lambda} \quad & \pi(p,\lambda) = \big(p - \lambda p_H - (1-\lambda) p_L\big)(w_H - w_L) \\ \text{s.t.} \quad & 0 < \lambda < 1, \qquad p < p_H \frac{\lambda\gamma}{\gamma - 1 + \lambda} + p_L \Big(1 - \frac{\lambda\gamma}{\gamma - 1 + \lambda}\Big) \end{aligned} \tag{9}$$

The following lemma provides a simple closed-form solution to this problem.

Lemma 1 *Given the prices of the two component goods* (p_H, p_L)*, the optimal design of the probabilistic good and the corresponding optimal profit gain are given by*

$$p^* = \frac{p_H + p_L}{2}, \qquad \lambda^* = \sqrt{\gamma(\gamma-1)} - \gamma + 1 = \frac{\sqrt{q_L}}{\sqrt{q_H} + \sqrt{q_L}}$$

$$\pi^*(p_H, p_L) = \Big(\frac{1}{2} - \lambda^*\Big) \frac{2\lambda^*\gamma - \gamma + 1 - \lambda^*}{2\lambda^*(1-\lambda^*)} \big(p_H - p_L\big)^2.$$

Different from the literature finding for horizontally differentiated markets (Fay and Xie 2008, Huang and Yu 2014) that equal-probability mixing is optimal, Lemma 1 gives an example where the optimal mixing probability is strictly below 0.5. Interestingly, the optimal price of the probabilistic good is the arithmetic mean of the prices of the two component goods, whereas the optimal quality of the probabilistic good is the geometric mean of the qualities of the component goods (i.e., $\lambda^* q_H + (1-\lambda^*) q_L = \sqrt{q_H q_L}$).

Given the closed-form expression of (p^*, λ^*), we can verify that the technical conditions for Proposition 5 are satisfied for the canonical utility function. Indeed, the condition of (8) holds because $\pi^*(p_H, p_L)$ is increasing (decreasing) in p_H (p_L) at any pair (p_H, p_L), not just at (p_h^*, p_l^*). For the condition of (7), note λ^* is not a function of p_H, hence, the condition becomes

$$\frac{\gamma}{1-\lambda^*}\Big(2 - \frac{1}{2} + c \cdot 0\Big) > \frac{1}{2} + \big(p^* - p_H + (1-\lambda^*)c\big) \frac{\partial^*}{\partial p_H}\Big(\frac{\gamma}{1-\lambda^*}\Big) = \frac{1}{2} + 0$$

which is true because $\gamma > 1$ and $\lambda^* < 1$. Therefore, if consumers have the canonical utility function, the price of the high-quality (low-quality) component good increases (decreases) after probabilistic selling is introduced. In fact, we can explicitly solve for the optimal prices of component goods, i.e., the optimization problem of (6). We summarize the optimal prices and the market implication in Proposition 6. Let $\tau \equiv \frac{q_L}{q_L - q_0} > 1$.

Proposition 6 *The optimal design of probabilistic selling is given by*

$$p_H^* = \frac{\kappa + 2\tau - 1 + \big((2\tau-1)\gamma + \kappa\big)c_H + \big(\kappa(\tau-1) - \gamma(2\tau-1)\big)c_L}{2\tau(\kappa+1) - 1}$$

$$p_L^* = \frac{\kappa + (\kappa - \gamma + 1)c_H + \big((\tau - 1)(\kappa + 1) + \gamma\big)c_L}{2\tau(\kappa + 1) - 1}$$

$$p^* = \frac{p_H^* + p_L^*}{2} = \frac{2\kappa + 2\tau - 1 + \big(2\kappa + 1 + 2\gamma(\tau - 1)\big)c_H + (\tau - 1)(2\kappa - 2\gamma + 1)c_L}{4\tau(\kappa + 1) - 2}$$

$$\lambda^* = \frac{\sqrt{q_L}}{\sqrt{q_H} + \sqrt{q_L}}$$

where $\kappa \equiv \frac{1}{2}(\sqrt{\gamma} - \sqrt{\gamma - 1})^{-2}$, *and the relative increase of market coverage is*

$$\frac{p_l^* - p_L^*}{1 - p_l^* \tau}\tau = \frac{2\gamma - \kappa - 1}{2\tau(\kappa + 1) - 1} \cdot \frac{2\tau - 1 - (1 - 2\gamma\tau)c_H - (2\gamma\tau + \tau - 1)c_L}{2\gamma\tau - 1 + \tau - \gamma\tau c_H - \gamma\tau(2\tau - 1)c_L}.$$

By viewing the probabilistic good as another "component" good, we can potentially design two additional probabilistic goods, with one mixing the high-quality component good and the probabilistic good, and the other mixing the low-quality component good and the probabilistic good. In fact, such a construction process can be repeated, which leads to the general questions of whether and how to design multiple probabilistic goods based on the two (original) component goods. To shed light on this interesting question, we take advantage of the analytical tractability of the canonical utility function to solve for the optimal menu of probabilistic goods.

Suppose the seller creates n probabilistic goods from the two component goods, H and L, with different mixing probabilities and hence different prices. Indexing these probabilistic goods by $i \in \{1, 2, ..., n\}$, we denote by λ_i the probability of receiving the high-quality component good when a consumer purchases the i-th probabilistic good, at price p_i. Without loss of generality, we assume $\lambda_1 > \cdots > \lambda_n$. For ease of notation, we also use $i = 0$ to index the high-quality component good H, and use $i = n + 1$ to index the low-quality component good L.

Denote the intersection of $q_i \cdot (w - p_i)$ with $q_{i-1} \cdot (w - p_{i-1})$ by $w_{i-1,i}$ — the budget level at which a consumer is indifferent between buying the $(i-1)$-th and the i-th probabilistic goods. We have

$$w_{i-1,i} = \frac{q_{i-1}p_{i-1} - q_i p_i}{q_{i-1} - q_i} = \frac{(\lambda_{i-1}(q_H - q_L) + q_L)p_{i-1} - (\lambda_i(q_H - q_L) + q_L)p_i}{(\lambda_{i-1} - \lambda_i)(q_H - q_L)}. \tag{10}$$

With the result of Lemma 1, we can derive a series of explicit relations of the optimal mixing probabilities as well as the optimal prices among all probabilistic goods. The key insight is that for any probabilistic good to be optimal, it must be the optimal single probabilistic good when it is considered as a probabilistic good constructed using its two neighboring goods in the sequence of synthetic or component goods. For any $i \in \{2, \cdots, n-1\}$, we can directly apply Lemma 1 to write λ_i in terms of λ_{i-1} and λ_{i+1}, and similarly, to write p_i in terms of p_{i-1} and p_{i+1}. Define $r = q_L/q_H \in (0, 1)$. The following lemma generalizes Lemma 1.

Lemma 2 *Given p_H, p_L, the optimal menu of n probabilistic goods consists of a sequence of probabilistic goods $\{\lambda_i, p_i\}$, where $\forall i \in \{1, \cdots, n\}$,*

$$p_i^* = \frac{n+1-i}{n+1} p_H + \frac{i}{n+1} p_L, \qquad \lambda_i^* = \frac{r^{\frac{i}{n+1}} - r}{1-r}. \tag{11}$$

Let $\Delta p \equiv p_H - p_L$. The optimal profit gain from selling n probabilistic goods, denoted by $\pi^*(n)$, is

$$\pi^*(n) = \frac{p_H \Delta p}{1-r} - \frac{\frac{\Delta p}{n+1}\left(\frac{\Delta p}{n+1} + p_L\right)}{1 - r^{\frac{1}{n+1}}} - \frac{n}{n+1}\left(\frac{\Delta p^2}{2} + p_L \Delta p + \Delta p^2 \frac{\frac{1}{n+1}}{1 - r^{\frac{1}{n+1}}}\right) + p_L \Delta p \left(\frac{\frac{1}{n+1}}{r^{-\frac{1}{n+1}} - 1} - \frac{r}{1-r}\right).$$

The result above suggests the optimal sequence of probabilistic goods is characterized by an arithmetic sequence of prices and a geometric sequence of qualities. That the price sequence forms an arithmetic sequence is clear from (11), which is rooted in Lemma 1, where we showed the price of the optimal single probabilistic good is the arithmetic mean of the prices of the component goods. To see why the qualities of probabilistic goods form a geometric sequence, note

$$q_i^* = \lambda_i^* q_H + (1-\lambda_i^*) q_L = q_H \cdot \left(\lambda_i^* + (1-\lambda_i^*) r\right) = q_H \cdot \left(\lambda_i^*(1-r) + r\right) = q_H r^{\frac{i}{n+1}},$$

where the last equality follows from (11). This geometric sequence characterization is again rooted in Lemma 1, where we showed the quality of the optimal single probabilistic good is the geometric mean of the qualities of the two component goods.

Finally, we consider the optimal pricing of component goods when multiple probabilistic goods are offered. The following proposition summarizes the result and includes Proposition 6 as a special case.

Proposition 7 (Optimal Menu Design of Probabilistic Selling) *The optimal design of probabilistic selling with n probabilistic goods is given by*

$$\begin{aligned}
p_H^* &= \frac{1}{\zeta(1+\frac{1}{2\tau-1}) + n - \frac{1}{2\tau-1}}\left(\frac{\zeta-1}{2\tau-1} + n + 1 + c\Big(\frac{n}{1-r} + \frac{\zeta}{2} - \frac{1-r^n}{(1-r)^2} r^{\frac{1}{n+1}}\Big) + c_H \frac{\zeta-1}{2\tau-1}\right) \\
p_L^* &= \frac{1 + c_H + (\tau-1)c_L}{2\tau-1} \\
&\quad - \frac{1}{(2\tau-1)\big(\zeta(1+\frac{1}{2\tau-1}) + n - \frac{1}{2\tau-1}\big)}\left(\frac{\zeta-1}{2\tau-1} + n + 1 + c\Big(\frac{n}{1-r} + \frac{\zeta}{2} - \frac{1-r^n}{(1-r)^2} r^{\frac{1}{n+1}}\Big) + c_H \frac{\zeta-1}{2\tau-1}\right) \\
\lambda_i^* &= \frac{r^{\frac{i}{n+1}} - r}{1-r} \\
p_i^* &= \frac{n+1-i}{n+1} p_H^* + \frac{i}{n+1} p_L^*
\end{aligned}$$

where $\zeta \equiv \frac{2}{1 - r^{\frac{1}{n+1}}}$.

It's clear from the solution that p_H^* and p_L^* satisfy the simple relation of

$$p_H^* + (2\tau - 1)p_L^* = 1 + c_H + (\tau - 1)c_L.$$

As expected, in the special case of a single probabilistic good (i.e., $n = 1$), the expressions for p_H^* and p_L^* reduce to their corresponding expressions in Proposition 6 by noting

$$\zeta = \frac{2}{1-\sqrt{r}} = 1 + \frac{1+\sqrt{r}}{1-\sqrt{r}} = 1 + \frac{1+\sqrt{\frac{\gamma-1}{\gamma}}}{1-\sqrt{\frac{\gamma-1}{\gamma}}} = 1 + \frac{\sqrt{\gamma}+\sqrt{\gamma-1}}{\sqrt{\gamma}-\sqrt{\gamma-1}} = 1 + \frac{1}{(\sqrt{\gamma}-\sqrt{\gamma-1})^2} = 1 + 2\kappa.$$

5. Conclusion

This paper studies two fundamental questions regarding probabilistic selling in vertically differentiated markets: When is it profitable and how does one design it optimally? For the first question, we identified convexity of consumer preference as an important factor and λ-concavity as a sufficient condition for probabilistic selling to be profitable. This insight helps us explain why the extant literature finds probabilistic selling is never profitable unless one introduces certain capacity constraint or bounded rationality. For the second question, we developed a theory of optimal probabilistic selling.

Our contributions to the literature are fourfold. First, we identified an important but overlooked driver of the benefit of probabilistic selling, which is rooted in consumer preference. Hence, probabilistic selling can be a profitable strategy in economic situations even in the absence of the factors suggested by the extant literature (Huang and Yu 2014, Zhang et al. 2015, Zheng et al. 2019). Second, we characterized some important structural properties of the optimal probabilistic selling strategy that should apply to many settings. In particular, we find probabilistic selling can increase market coverage and economic efficiency. Third, we initiated the study of designing multiple probabilistic goods, which, given the exclusive focus on the design of single probabilistic good in the extant literature and the call for attention to this question by Zhang et al. (2015), is both theoretically interesting and practically relevant. Although we characterized the optimal design of multiple probabilistic goods based on a specific utility function that is often used in the economics literature on product vertical differentiation, our approach to reduce the dimensionality of the optimization problem for the design of multiple probabilistic goods should be applicable when other types of utility functions are considered. Fourth, the drastically different finding obtained from strictly convex consumer preference suggests linear approximation is not always without consequence. Analytical research can sometimes benefit from a robustness check with some alternative utility functions.

The paper also has important managerial implications for practitioners. First, because preference convexity is a widely accepted notion in economics about consumer preference, we believe the potential of probabilistic selling is beyond what has been discussed in the current literature. In the absence of the administrative cost of selling synthetic goods, probabilistic selling can be a generic pricing strategy profitable for a variety of products. Through online selling or technological innovation, the administrative cost of probabilistic selling can be made negligible in the future. Indeed, some innovative company may design and offer a common platform for all sellers who are interested in probabilistic selling, thereby driving down the administrative cost. Second, because the profit gain from probabilistic selling should increase as the quality (or price) difference between two component goods increases, probabilistic selling is particularly appealing in market settings where the quality (or price) difference between different goods is substantial. For example, in the airline industry, the quality and price difference between "first class" and "economy class" is large, hence, popular marketing strategies such as "elite membership" or upgradable tickets, which can be interpreted as forms of probabilistic selling, are likely profitable for airlines. Similarly, in the hotel industry, the quality and price difference between hotels of different star ratings or rooms of different sizes and amenities can be significant; hence, marketing strategies such as Delphina's Formula Roulette Prestige program or H10's Tenerife Roulette program are attractive to consumers who dislike extreme budge allocation between quality and money. On the other hand, for markets where product quality differences are small and administrative costs are large, probabilistic selling may not be profitable in practice. Third, a key result from our theory of optimal probabilistic selling is that under some technical conditions, the market coverage increases as a result of probabilistic selling. In other words, fewer consumers will be priced out of the market. Therefore, the practice of probabilistic selling not only improves profit, but also increases economic efficiency. This takeaway is an important one for policy-makers as well as for sellers who have a strategic interest in market penetration. Finally, our theory of optimal probabilistic selling also provides direct guidance to practitioners when they implement the strategy.

The current paper has several limitations that are worth future exploration. First, although the concept of λ-concavity is useful, many open questions remain. For example, for any strictly quasi-concave utility function satisfying the regularity conditions and any distribution, does there exist a $\lambda \in (0,1)$ such that the pair is λ-concave? If not, what are the requirements for the utility function and/or the distribution? Second, due to its abstract nature, the technical conditions of Proposition 5 may be difficult to interpret and apply in practice. Further characterization and generalization would be theoretically interesting and practically valuable. Third, we abstracted away from the quality design of the component goods. For certain industries, exploring the implication of probabilistic selling on the optimal quality design of component goods might be interesting. Fourth, we did not consider competition among multiple sellers, which could be another interesting direction for future research.

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Appendix

A. Main Proofs

To prove the lemmas and propositions in the main text, we need the following lemma first.

Lemma A.1 *Let $D_H(w)$ be the utility difference between buying a high-quality component good and buying a probabilistic good; let $D_L(w)$ be the utility difference between buying a low-quality component good and buying a probabilistic good; and let $D(w)$ be the utility difference between buying a high-quality component good and buying a low-quality component good.*

$$\begin{aligned} D_H(w) &\equiv U(w-p_H,q_H)-\lambda U(w-p,q_H)-(1-\lambda)U(w-p,q_L) \\ D_L(w) &\equiv U(w-p_L,q_L)-\lambda U(w-p,q_H)-(1-\lambda)U(w-p,q_L) \\ D(w) &\equiv U(w-p_H,q_H)-U(w-p_L,q_L). \end{aligned}$$

Then, $D(w)$ is strictly increasing in w, and $\forall\lambda\in(0,1)$ and $\forall p\in(p_L,p_H)$,

- *$D_H(w)$ is strictly increasing in w;*
- *$D_L(w)$ is strictly decreasing in w.*

Proof of Lemma A.1.

For $D(w)$, we have

$$D(w)=\Big(U(w-p_H,q_H)-U(w-p_H,q_L)\Big)-\Big(U(w-p_L,q_L)-U(w-p_H,q_L)\Big)$$

where the term in the first parentheses is strictly increasing in w thanks to the single-crossing condition, whereas the term in the second parentheses is (weakly) decreasing in w because $\partial^2U/\partial x^2\leq 0$. Hence, $D(w)$ is strictly increasing in w.

For $D_H(w)$, we have

$$\begin{aligned} D_H(w) &= U(w-p_H,q_H)-\lambda U(w-p,q_H)-(1-\lambda)U(w-p,q_L) \\ &= (1-\lambda)\Big(U(w-p_H,q_H)-U(w-p,q_L)\Big)-\lambda\Big(U(w-p,q_H)-U(w-p_H,q_H)\Big) \\ &= (1-\lambda)\Big(U(w-p_H,q_H)-U(w-p_H,q_L)\Big)+(1-\lambda)\Big(U(w-p_H,q_L)-U(w-p,q_L)\Big) \\ &\quad -\lambda\Big(U(w-p,q_H)-U(w-p_H,q_H)\Big) \\ &= (1-\lambda)\Big(U(w-p_H,q_H)-U(w-p_H,q_L)\Big)-(1-\lambda)\Big(U(w-p,q_L)-U(w-p_H,q_L)\Big) \\ &\quad -\lambda\Big(U(w-p,q_H)-U(w-p_H,q_H)\Big) \end{aligned}$$

The term in the first parentheses is strictly increasing in w thanks to the single-crossing condition, whereas the term in the second parentheses and the term in the third parentheses are both (weakly) decreasing in w because $\partial^2U/\partial x^2\leq 0$. Hence, $D_H(w)$ is strictly increasing in w.

Similarly, for $D_L(w)$, we have

$$\begin{aligned} D_L(w) &= U(w-p_L,q_L) - \lambda U(w-p,q_H) - (1-\lambda)U(w-p,q_L) \\ &= \lambda\big(U(w-p_L,q_L) - U(w-p,q_H)\big) + (1-\lambda)\Big(U(w-p_L,q_L) - U(w-p,q_L)\Big) \\ &= \lambda\Big(U(w-p_L,q_H) - U(w-p,q_H)\Big) - \lambda\Big(U(w-p_L,q_H) - U(w-p_L,q_L)\Big) \\ &\quad +(1-\lambda)\Big(U(w-p_L,q_L) - U(w-p,q_L)\Big) \end{aligned}$$

where the term in the first parentheses and the term in the third parentheses are both (weakly) decreasing because $\partial^2 U/\partial x^2 \leq 0$ and the term in the second parentheses is strictly increasing in w thanks to the single-crossing condition. Hence, $D_L(w)$ is strictly decreasing in w.

■.

Proof of Proposition 1.

To see the "if" part, note that by the definition of $\bar{p}$, we have, for any $p < \bar{p}$,

$$\begin{aligned} & \lambda U(w_0-p,q_H) + (1-\lambda)U(w_0-p,q_L) - U(w_0-p_H,q_H) \\ > \quad & \lambda U(w_0-\bar{p},q_H) + (1-\lambda)U(w_0-\bar{p},q_L) - U(w_0-p_H,q_H) = 0 \\ & \lambda U(w_0-p,q_H) + (1-\lambda)U(w_0-p,q_L) - U(w_0-p_L,q_L) \\ > \quad & \lambda U(w_0-\bar{p},q_H) + (1-\lambda)U(w_0-\bar{p},q_L) - U(w_0-p_L,q_L) = 0. \end{aligned}$$

In other words, consumers with budget w_0 strictly prefer the probabilistic good to either component good. By the continuity of U, there exists a neighborhood $(w_0-\epsilon, w_0+\epsilon)$ such that consumers with budget $w \in (w_0-\epsilon, w_0+\epsilon)$ all strictly prefer the probabilistic good to either component good. By the absolute continuity of the budget distribution, the demand for the probabilistic good is positive.

To show the "only if" part, we show its contrapositive statement: there is no demand for the probabilistic good if $p \geq \bar{p}$. It suffices to consider the following cases.

1. For consumers with $w > w_0$, by Lemma A.1, we have

$$\begin{aligned} & U(w-p_H,q_H) - \lambda U(w-p,q_H) - (1-\lambda)U(w-p,q_L) \\ > \; & U(w_0-p_H,q_H) - \lambda U(w_0-p,q_H) - (1-\lambda)U(w_0-p,q_L) \\ \geq \; & U(w_0-p_H,q_H) - \lambda U(w_0-\bar{p},q_H) - (1-\lambda)U(w_0-\bar{p},q_L) \\ = \; & 0 \end{aligned}$$

which implies that these consumers prefer the high-quality good to the probabilistic good.

2. For consumers with $w < w_0$, by Lemma A.1, we have

$$U(w-p_L,q_L) - \lambda U(w-p,q_H) - (1-\lambda)U(w-p,q_L)$$

$$
\begin{aligned}
&> U(w_0 - p_L, q_L) - \lambda U(w_0 - p, q_H) - (1-\lambda)U(w_0 - p, q_L) \\
&\geq U(w_0 - p_L, q_L) - \lambda U(w_0 - \bar{p}, q_H) - (1-\lambda)U(w_0 - \bar{p}, q_L) \\
&= 0
\end{aligned}
$$

which implies that these consumers prefer the low-quality good to the probabilistic good.
By the absolute continuity of the budget distribution, the demand for the probabilistic good is zero.

We now show the equivalent characterization of positive demand via the pivotal consumer. By definition of $\bar{p}$ in (3), we have

$$\lambda U(w_0 - \bar{p}, q_H) + (1-\lambda)U(w_0 - \bar{p}, q_L) = U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L)$$

Because $\lambda U(w_0 - p, q_H) + (1-\lambda)U(w_0 - p, q_L)$ is strictly decreasing in p, the condition of $p < \bar{p}$ is equivalent to the following:

$$\lambda U(w_0 - p, q_H) + (1-\lambda)U(w_0 - p, q_L) > \lambda U(w_0 - \bar{p}, q_H) + (1-\lambda)U(w_0 - \bar{p}, q_L) = U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L).$$

But the left-hand-side is the expected utility of purchasing a probabilistic good for a pivotal consumer. Hence, a probabilistic good has positive demand if and only if a pivotal consumer strictly prefers it to any component good.

To prove $w_L < w_0 < w_H$, we rewrite $D_H(w)$ and $D_L(w)$ in Lemma A.1 as functions of both w and p:

$$
\begin{aligned}
D_H(w,p) &\equiv U(w - p_H, q_H) - \lambda U(w - p, q_H) - (1-\lambda)U(w - p, q_L) \\
D_L(w,p) &\equiv U(w - p_L, q_L) - \lambda U(w - p, q_H) - (1-\lambda)U(w - p, q_L).
\end{aligned}
$$

By the definition of $\bar{p}$, we have

$$D_H(w_0, \bar{p}) = D_L(w_0, \bar{p}) = 0.$$

By the definitions of w_H and w_L, for any $p < \bar{p}$,

$$D_H(w_H, p) = D_L(w_L, p) = 0.$$

Hence, we have

$$D_H(w_H, \bar{p}) > D_H(w_H, p) = 0 = D_H(w_0, \bar{p}) \quad \text{and} \quad D_L(w_L, \bar{p}) > D_L(w_L, p) = 0 = D_L(w_0, \bar{p}).$$

By Lemma A.1, $D_H(w,p)$ is strictly increasing in w and $D_L(w,p)$ is strictly decreasing in w, hence we have $w_H > w_0 > w_L$.

Finally, we show $\bar{p} > \lambda p_H + (1-\lambda)p_L$ which is equivalent to the following condition by the definition of $\bar{p}$ in (3),

$$\lambda U(w_0 - \lambda p_H - (1-\lambda)p_L, q_H) + (1-\lambda)U(w_0 - \lambda p_H - (1-\lambda)p_L, q_L) > U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L).$$

Denote $f(\lambda) = \lambda U(w_0 - \lambda p_H - (1-\lambda)p_L, q_H) + (1-\lambda)U(w_0 - \lambda p_H - (1-\lambda)p_L, q_L)$. First, we note

$$f(0) = f(1) = U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L).$$

It suffices to show $f(\lambda)$ is strictly concave.

$$\begin{aligned} f'(\lambda) = \; & U(w_0 - \lambda p_H - (1-\lambda)p_L, q_H) && + \lambda U_1(w_0 - \lambda p_H - (1-\lambda)p_L, q_H)(p_L - p_H) \\ & -U(w_0 - \lambda p_H - (1-\lambda)p_L, q_L) && + (1-\lambda)U_1(w_0 - \lambda p_H - (1-\lambda)p_L, q_L)(p_L - p_H) \\ \frac{f''(\lambda)}{p_L - p_H} = \; & 2U_1(w_0 - \lambda p_H - (1-\lambda)p_L, q_H) && + \lambda U_{11}(w_0 - \lambda p_H - (1-\lambda)p_L, q_H)(p_L - p_H) \\ & -2U_1(w_0 - \lambda p_H - (1-\lambda)p_L, q_L) && + (1-\lambda)U_{11}(w_0 - \lambda p_H - (1-\lambda)p_L, q_L)(p_L - p_H) \end{aligned}$$

Hence, $f''(\lambda) < 0$ is equivalent to

$$\begin{aligned} & 2\Big(U_1(w_0 - \lambda p_H - (1-\lambda)p_L, q_H) - U_1(w_0 - \lambda p_H - (1-\lambda)p_L, q_L)\Big) \\ > \; & (p_H - p_L)\Big(\lambda U_{11}(w_0 - \lambda p_H - (1-\lambda)p_L, q_H) + (1-\lambda)U_{11}(w_0 - \lambda p_H - (1-\lambda)p_L, q_L)\Big) \end{aligned}$$

which is obvious because the left-hand-side is positive by the single-crossing condition and the right-hand-side is non-positive because $U_{11} \le 0$.

■.

Proof of Proposition 2

Because the seller does not have extra capacity, given the prices of the component goods (p_H, p_L), probabilistic selling is profitable only if the revenue from the probabilistic goods exceeds the cannibalized revenue from selling those component goods that consist of the probabilistic good. Hence, we must have $p > \lambda p_H + (1-\lambda)p_L$ where the right-hand-side is the average unit revenue of the component goods. Equivalently, we have $w_0 - p < w_0 - \lambda p_H - (1-\lambda)p_L = \lambda(w_0 - p_H) + (1-\lambda)(w_0 - p_L)$.

$$\begin{aligned} U\big(\lambda(w_0 - p_H) + (1-\lambda)(w_0 - p_L), \lambda q_H + (1-\lambda)q_L\big) &\ge U(w_0 - p, \lambda q_H + (1-\lambda)q_L) \\ &\ge \lambda U(w_0 - p, q_H) + (1-\lambda)U(w_0 - p, q_L) \\ &> \lambda U(w_0 - \bar{p}, q_H) + (1-\lambda)U(w_0 - \bar{p}, q_L) \\ &= \lambda U(w_0 - p_H, q_H) + (1-\lambda)U(w_0 - p_L, q_L). \end{aligned}$$

where the first inequality is because $U(w,q)$ increases in w; the second inequality follows from the concavity of U with respect to quality improvement (i.e., $\partial U^2/\partial y^2 \leq 0$); the third inequality is because $p<\bar{p}$ by Proposition 1; and the last equality follows from the definition of $\bar{p}$:

$$\lambda U(w_0-\bar{p},q_H)+(1-\lambda)U(w_0-\bar{p},q_L)=U(w_0-p_H,q_H)=U(w_0-p_L,q_L).$$

So, $U\big(\lambda(w_0-p_H)+(1-\lambda)(w_0-p_L),\lambda q_H+(1-\lambda)q_L\big)>\lambda U(w_0-p_H,q_H)+(1-\lambda)U(w_0-p_L,q_L)$.

Because the above inequality holds for any p_H, q_H, p_L, and q_L, we conclude that $\forall\lambda\in(0,1)$, and any two distinct bundles (w_1,q_1) and (w_2,q_2) on the same indifference curve,

$$U\big(\lambda(w_1,q_1)+(1-\lambda)(w_2,q_2)\big)>U(w_1,q_1)=U(w_2,q_2). \tag{A.1}$$

The above is an equivalent definition of a strictly quasiconcave utility function when the utility function is increasing in all attributes. Indeed, let U be increasing in each dimension. We want to show U is (strictly) quasiconcave if and only if its indifference curves are (strictly) convex. The "only if" part is obvious from the definition of (strict) quasiconcavity. For the "if" part, suppose $U(x)>U(y)$, then by reducing x_i for those i where $x_i>y_i$, we can find a $\tilde{x}$ such that $u(\tilde{x})=u(y)$. Hence,

$$U\big(\lambda x+(1-\lambda)y\big)\geq U\big(\lambda\tilde{x}+(1-\lambda)y\big)\geq U(y).$$

The proof is complete by noting the equivalence between strictly convex preference and strictly quasiconcave utility. ■.

Proof of Proposition 3

Suppose (p_H,p_L) are set optimally by the seller to maximize the profit in the absence of probabilistic selling, i.e., $(p_H,p_L)=(p_h^*,p_l^*)$. Consider a candidate probabilistic good (p,λ) where $p=\lambda p_H+(1-\lambda)p_L$. From Proposition 1, we know $\bar{p}>p$, hence there is a positive demand for the probabilistic good. To show the new profit from selling the probabilistic good to consumers with budget $w\in[w_L,w_H]$ exceeds the lost profit from selling component goods to these same consumers, we compare the profit from selling $(p_H,p_L,\lambda p_H+(1-\lambda)p_L,\lambda)$ with the profit from selling (p_H,p_L).

Because the seller collects the same amount of profit from consumers with budgets $w\leq w_L$ or $w\geq w_H$ under the two selling strategies, the profit gain from probabilistic selling is

$$\begin{aligned}\pi &= \big(p-(\lambda c_H+(1-\lambda)c_L)\big)(F(w_H)-F(w_L))-(p_H-c_H)(F(w_H)-F(w_0))-(p_L-c_L)(F(w_0)-F(w_L))\\ &= \big(p-\lambda p_H-(1-\lambda)p_L\big)(F(w_H)-F(w_L))+(p_H-c_H-p_L+c_L)\big((F(w_0)-F(w_L))\lambda-(F(w_H)-F(w_0))(1-\lambda)\big)\\ &= (p_H-c_H-p_L+c_L)\big((F(w_0)-F(w_L))\lambda-(F(w_H)-F(w_0))(1-\lambda)\big)\end{aligned}$$

First, we show $p_H - c_H - p_L + c_L > 0$ when (p_H, p_L) are set optimally by the seller in the absence of probabilistic selling. Because the objective function is

$$\pi^0(p_H, p_L) = (p_H - c_H)\big(1 - F(w_0)\big) + (p_L - c_L)\big(F(w_0) - F(\underline{w})\big),$$

the first-order condition with respect to p_H is

$$\begin{aligned} & 1 - F(w_0) - f(w_0)(p_H - c_H)\frac{\partial w_0}{\partial p_H} + (p_L - c_L) f(w_0)\frac{\partial w_0}{\partial p_H} = 0 \\ \Longleftrightarrow \quad & \frac{\partial w_0}{\partial p_H} = \frac{1 - F(w_0)}{f(w_0)(p_H - c_H - p_L + c_L)} \end{aligned}$$

w_0 is an implicit function of p_H and p_L determined by $U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L)$. Taking derivative with respect to p_H, we obtain

$$\frac{\partial w_0}{\partial p_H} = \frac{U_1(w_0 - p_H, q_H)}{U_1(w_0 - p_H, q_H) - U_1(w_0 - p_L, q_L)} > 1$$

where the inequality follows from Lemma A.1. Therefore, we must have $p_H - c_H - p_L + c_L > 0$. Note this inequality implies that at the optimal prices (without probabilistic selling) p_h^* and p_l^*, selling the probabilistic good $(\lambda p_h^* + (1-\lambda)p_l^*, \lambda)$ yields higher profit margin in the segment (w_L, w_0) but lower profit margin in the segment (w_0, w_H), compared with not selling the probabilistic good, because

$$\begin{aligned} p_H - c_H - \big[\lambda p_H + (1-\lambda)p_L - \lambda c_H - (1-\lambda)c_L\big] &= (1-\lambda)(p_H - c_H - p_L + c_L) > 0 \\ \big[\lambda p_H + (1-\lambda)p_L - \lambda c_H - (1-\lambda)c_L\big] - (p_L - c_L) &= \lambda(p_H - c_H - p_L + c_L) > 0. \end{aligned}$$

It remains to note that $\pi > 0$ if $(F(w_0) - F(w_L))\lambda > (F(w_H) - F(w_0))(1-\lambda)$ for some $\lambda \in (0,1)$, or equivalently

$$F(w_0) > \lambda F(w_L) + (1-\lambda)F(w_H).$$

The above is the condition of λ-concavity once we recognize $w_L = x$, $w_0 = y$, and $w_H = z$ for this candidate probabilistic selling strategy.

■.

Proof of Example 2

By the definitions of w_H, w_L, w_0 and for Cobb-Douglas utility, we have

$$\begin{aligned} (w_H - p_H)^{\frac{1}{\alpha}} q_H &= (w_H - p)^{\frac{1}{\alpha}} \big(\lambda q_H + (1-\lambda) q_L\big) \\ (w_L - p_L)^{\frac{1}{\alpha}} q_L &= (w_L - p)^{\frac{1}{\alpha}} \big(\lambda q_H + (1-\lambda) q_L\big) \end{aligned}$$

Solving these equations, we obtain w_H, w_L, w_0 as functions of p, λ,

$$w_H = p + \frac{q_H^\alpha (p_H - p)}{q_H^\alpha - (\lambda q_H + (1-\lambda)q_L)^\alpha}, \quad w_L = p + \frac{q_L^\alpha (p - p_L)}{(\lambda q_H + (1-\lambda)q_L)^\alpha - q_L^\alpha}$$

Recall

$$w_0 = \frac{q_H^\alpha p_H - q_L^\alpha p_L}{q_H^\alpha - q_L^\alpha} = p + \frac{q_H^\alpha (p_H - p) + q_L^\alpha (p - p_L)}{q_H^\alpha - q_L^\alpha}.$$

With uniform distribution, the condition of λ-concavity simplifies to

$$\frac{q_H^\alpha (p_H - p) + q_L^\alpha (p - p_L)}{q_H^\alpha - q_L^\alpha} > \frac{\lambda q_L^\alpha (p - p_L)}{(\lambda q_H + (1-\lambda) q_L)^\alpha - q_L^\alpha} + \frac{(1-\lambda) q_H^\alpha (p_H - p)}{q_H^\alpha - (\lambda q_H + (1-\lambda) q_L)^\alpha}.$$

Because $p = \lambda p_H + (1-\lambda) p_L \Rightarrow \lambda(p_H - p) = (1-\lambda)(p - p_L)$, we can rewrite the above inequality as

$$\frac{(1-\lambda) q_H^\alpha + \lambda q_L^\alpha}{q_H^\alpha - q_L^\alpha} > \frac{\lambda^2 q_L^\alpha}{(\lambda q_H + (1-\lambda) q_L)^\alpha - q_L^\alpha} + \frac{(1-\lambda)^2 q_H^\alpha}{q_H^\alpha - (\lambda q_H + (1-\lambda) q_L)^\alpha}.$$

Denote $h = q_H^\alpha$, $l = q_L^\alpha$, $m = (\lambda q_H + (1-\lambda) q_L)^\alpha$, the above inequality simplifies to

$$\begin{aligned} & \lambda l + (1-\lambda) h > \lambda^2 l \Big(1 + \frac{h-m}{m-l}\Big) + (1-\lambda)^2 h \Big(1 + \frac{m-l}{h-m}\Big) \\ \iff & \lambda(1-\lambda)(h+l) > \lambda^2 l \frac{h-m}{m-l} + (1-\lambda)^2 h \frac{m-l}{h-m} \end{aligned}$$

Pick $\lambda = 1/2$. The above inequality becomes

$$h \frac{h - 2m + l}{h - m} > l \frac{h - 2m + l}{m - l} \iff \frac{h}{h-m} > \frac{l}{m-l} \iff m^{-1} \le \frac{h^{-1} + l^{-1}}{2}$$

where the first step is because $h - 2m + l > 0$ by Jensen's inequality for $x \mapsto x^\alpha$ and the last inequality is by Jensen's inequality for $x \mapsto x^{-\alpha}$. ■.

Proof of Proposition 4

Take derivative of $U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L)$ on both sides with respect to p_H and p_L:

$$\frac{\partial w_0}{\partial p_H} = \frac{U_1(w_0 - p_H, q_H)}{U_1(w_0 - p_H, q_H) - U_1(w_0 - p_L, q_L)} > 1, \quad \frac{\partial w_0}{\partial p_L} = \frac{-U_1(w_0 - p_L, q_L)}{U_1(w_0 - p_H, q_H) - U_1(w_0 - p_L, q_L)} < 0.$$

where the inequalities follow from Lemma A.1. Therefore,

$$\frac{\partial w_0}{\partial p_H} + \frac{\partial w_0}{\partial p_L} = 1. \tag{A.2}$$

Similarly, recall the definitions of w_H, w_L:

$$U(w_H - p_H, q_H) = \lambda U(w_H - p, q_H) + (1-\lambda) U(w_H - p, q_L) \tag{A.3}$$

$$U(w_L - p_L, q_L) = \lambda U(w_L - p, q_H) + (1-\lambda) U(w_L - p, q_L). \tag{A.4}$$

$$\begin{aligned} U_1(w_H - p_H, q_H) \frac{\partial w_H}{\partial p} &= \lambda U_1(w_H - p, q_H) \Big(\frac{\partial w_H}{\partial p} - 1\Big) + (1-\lambda) U_1(w_H - p, q_L) \Big(\frac{\partial w_H}{\partial p} - 1\Big) \\ U_1(w_H - p_H, q_H) \frac{\partial w_H}{\partial \lambda} &= U(w_H - p, q_H) - U(w_H - p, q_L) + \Big(\lambda U_1(w_H - p, q_H) + (1-\lambda) U_1(w_H - p, q_L)\Big) \frac{\partial w_H}{\partial \lambda} \\ U_1(w_H - p_H, q_H) \Big(\frac{\partial w_H}{\partial p_H} - 1\Big) &= \lambda U_1(w_H - p, q_H) \frac{\partial w_H}{\partial p_H} + (1-\lambda) U_1(w_H - p, q_L) \frac{\partial w_H}{\partial p_H} \end{aligned}$$

$$
\begin{aligned}
U_1(w_L-p_L,q_L)\frac{\partial w_L}{\partial p} &= \lambda U_1(w_L-p,q_H)\Big(\frac{\partial w_L}{\partial p}-1\Big)+(1-\lambda)U_1(w_L-p,q_L)\Big(\frac{\partial w_L}{\partial p}-1\Big)\\
U_1(w_L-p_L,q_L)\frac{\partial w_L}{\partial \lambda} &= U(w_L-p,q_H)-U(w_L-p,q_L)+\Big(\lambda U_1(w_L-p,q_H)+(1-\lambda)U_1(w_L-p,q_L)\Big)\frac{\partial w_L}{\partial \lambda}\\
U_1(w_L-p_L,q_L)\Big(\frac{\partial w_L}{\partial p_L}-1\Big) &= \lambda U_1(w_L-p,q_H)\frac{\partial w_L}{\partial p_L}+(1-\lambda)U_1(w_L-p,q_L)\frac{\partial w_L}{\partial p_L}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial w_H}{\partial p} &= 1-\frac{U_1(w_H-p_H,q_H)}{U_1(w_H-p_H,q_H)-\lambda U_1(w_H-p,q_H)-(1-\lambda)U_1(w_H-p,q_L)}<0 && \text{(A.5)}\\
\frac{\partial w_H}{\partial \lambda} &= \frac{U(w_H-p,q_H)-U(w_H-p,q_L)}{U_1(w_H-p_H,q_H)-\lambda U_1(w_H-p,q_H)-(1-\lambda)U_1(w_H-p,q_L)}>0 && \text{(A.6)}\\
\frac{\partial w_H}{\partial p_H} &= \frac{U_1(w_H-p_H,q_H)}{U_1(w_H-p_H,q_H)-\lambda U_1(w_H-p,q_H)-(1-\lambda)U_1(w_H-p,q_L)}>0 && \text{(A.7)}\\
\frac{\partial w_L}{\partial p} &= 1-\frac{U_1(w_L-p_L,q_L)}{U_1(w_L-p_L,q_L)-\lambda U_1(w_L-p,q_H)-(1-\lambda)U_1(w_L-p,q_L)}>0 && \text{(A.8)}\\
\frac{\partial w_L}{\partial \lambda} &= \frac{U(w_L-p,q_H)-U(w_L-p,q_L)}{U_1(w_L-p_L,q_L)-\lambda U_1(w_L-p,q_H)-(1-\lambda)U_1(w_L-p,q_L)}<0 && \text{(A.9)}\\
\frac{\partial w_L}{\partial p_L} &= \frac{U_1(w_L-p_L,q_L)}{U_1(w_L-p_L,q_L)-\lambda U_1(w_L-p,q_H)-(1-\lambda)U_1(w_L-p,q_L)}<0 && \text{(A.10)}
\end{aligned}
$$

where the inequalities are due to Lemma A.1. From the expressions above, we have

$$
\frac{\partial w_H}{\partial p}+\frac{\partial w_H}{\partial p_H}=1 \quad \text{and} \quad \frac{\partial w_L}{\partial p}+\frac{\partial w_L}{\partial p_L}=1 \qquad \text{(A.11)}
$$

The profit gain from probabilistic selling is

$$
\begin{aligned}
\pi &= \big(p-\lambda c_H-(1-\lambda)c_L\big)(w_H-w_L)-(p_H-c_H)(w_H-w_0)-(p_L-c_L)(w_0-w_L)\\
&= \big(p-\lambda c_H-(1-\lambda)c_L-p_H+c_H\big)(w_H-w_0)+\big(p-\lambda c_H-(1-\lambda)c_L-p_L+c_L\big)(w_0-w_L)\\
&= \big(p-\lambda p_H-(1-\lambda)p_H+(1-\lambda)c\big)(w_H-w_0)+\big(p-(1-\lambda)p_L-\lambda p_L-\lambda c\big)(w_0-w_L)\\
&= \big(p-\lambda p_H-(1-\lambda)p_L\big)(w_H-w_L)+(p_H-p_L-c)\big(\lambda(w_0-w_L)-(1-\lambda)(w_H-w_0)\big)\\
&= \big(p-\lambda p_H-(1-\lambda)p_L\big)(w_H-w_L)+(p_H-p_L-c)\big(w_0-\lambda w_L-(1-\lambda)w_H\big)
\end{aligned}
$$

By the first-order condition, the optimal solution (p^*,λ^*) satisfies

$$
\begin{aligned}
w_H-w_L+\big(p-\lambda p_H-(1-\lambda)p_L\big)\Big(\frac{\partial w_H}{\partial p}-\frac{\partial w_L}{\partial p}\Big)&\\
-(p_H-p_L-c)\big(\lambda\frac{\partial w_L}{\partial p}+(1-\lambda)\frac{\partial w_H}{\partial p}\big) &= 0 && \text{(A.12)}\\
-(p_H-p_L)(w_H-w_L)+\big(p-\lambda p_H-(1-\lambda)p_L\big)\Big(\frac{\partial w_H}{\partial \lambda}-\frac{\partial w_L}{\partial \lambda}\Big)&\\
+(p_H-p_L-c)\big(w_H-w_L-\lambda\frac{\partial w_L}{\partial \lambda}+(1-\lambda)\frac{\partial w_H}{\partial \lambda}\big) &= 0 && \text{(A.13)}
\end{aligned}
$$

By the Envelope Theorem,

$$
\frac{\partial \pi^*(p_H,p_L)}{\partial p_H} = -\lambda(w_H-w_L)+\big(p-\lambda p_H-(1-\lambda)p_L\big)\frac{\partial w_H}{\partial p_H}+(w_0-w_L)\lambda-(w_H-w_0)(1-\lambda)
$$

$$+(p_H - p_L - c)\Big(\frac{\partial w_0}{\partial p_H} - (1-\lambda)\frac{\partial w_H}{\partial p_H}\Big) \tag{A.14}$$

$$= w_0 - w_H + \big(p - \lambda p_H - (1-\lambda)p_L\big)\frac{\partial w_H}{\partial p_H} + (p_H - p_L - c)\Big(\frac{\partial w_0}{\partial p_H} - (1-\lambda)\frac{\partial w_H}{\partial p_H}\Big)$$

$$\frac{\partial \pi^*(p_H,p_L)}{\partial p_L} = -(1-\lambda)(w_H - w_L) - \big(p - \lambda p_H - (1-\lambda)p_L\big)\frac{\partial w_L}{\partial p_L} + (w_H - w_0)(1-\lambda) - (w_0 - w_L)\lambda$$

$$+(p_H - p_L - c)\Big(\frac{\partial w_0}{\partial p_L} - \lambda\frac{\partial w_L}{\partial p_L}\Big) \tag{A.15}$$

$$= w_L - w_0 - \big(p - \lambda p_H - (1-\lambda)p_L\big)\frac{\partial w_L}{\partial p_L} + p_H - p_L - c)\Big(\frac{\partial w_0}{\partial p_L} - \lambda\frac{\partial w_L}{\partial p_L}\Big)$$

Thus,

$$\begin{aligned}\frac{\partial \pi^*(p_H,p_L)}{\partial p_H} + \frac{\partial \pi^*(p_H,p_L)}{\partial p_L} &= w_L - w_H \;+\; \big(p - \lambda p_H - (1-\lambda)p_L\big)\Big(\frac{\partial w_H}{\partial p_H} - \frac{\partial w_L}{\partial p_L}\Big) \\ &\quad + \;(p_H - p_L - c)\Big(\frac{\partial w_0}{\partial p_H} + \frac{\partial w_0}{\partial p_L} - \lambda\frac{\partial w_L}{\partial p_L} - (1-\lambda)\frac{\partial w_H}{\partial p_H}\Big) \\ &= w_L - w_H \;+\; \big(p - \lambda p_H - (1-\lambda)p_L\big)\Big(\frac{\partial w_L}{\partial p} - \frac{\partial w_H}{\partial p}\Big) \\ &\quad + \;(p_H - p_L - c)\Big(1 - \lambda\frac{\partial w_L}{\partial p_L} - (1-\lambda)\frac{\partial w_H}{\partial p_H}\Big)\end{aligned}$$

where the second equality is by (A.11) and (A.2). Hence, by (A.12), we have

$$\begin{aligned}\frac{\partial \pi^*(p_H,p_L)}{\partial p_H} + \frac{\partial \pi^*(p_H,p_L)}{\partial p_L} &= \;(p_H - p_L - c)\Big(1 - \lambda\frac{\partial w_L}{\partial p_L} - (1-\lambda)\frac{\partial w_H}{\partial p_H} - \lambda\frac{\partial w_L}{\partial p} - (1-\lambda)\frac{\partial w_H}{\partial p}\Big) \\ &= \;(p_H - p_L - c)\big(1 - \lambda - (1-\lambda)\big) = 0\end{aligned}$$

■.

Lemma A.2 *Let $F(x,y):\mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable function such that*

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x \partial y} = -a, \qquad \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} + b$$

where $a > 0$ and $b \neq -a$ are both constants. Let $t^* = a/(a+b)$. Suppose (x_0, y_0) is a point such that

$$\frac{\partial F}{\partial x}\Big|_{(x_0,y_0)} + \frac{\partial F}{\partial y}\Big|_{(x_0,y_0)} = 0.$$

Then, there exists a point (x^*, y^*) which is not necessarily unique, such that

$$\frac{\partial F}{\partial x}\Big|_{(x^*,y^*)} = \frac{\partial F}{\partial y}\Big|_{(x^*,y^*)} = 0$$

if any of the following is true.

1.

$$\frac{\partial F}{\partial x}\Big|_{(x_0,y_0)} > 0, \qquad \inf_{x\ge 0}\Big\{a + (1+t^*)\frac{\partial^2 F}{\partial x \partial y}\Big|_{(x,-t^*x)}\Big\} > 0,$$

$$\textit{or} \quad \frac{\partial F}{\partial x}\Big|_{(x_0,y_0)} < 0, \qquad \sup_{x\ge 0}\Big\{a + (1+t^*)\frac{\partial^2 F}{\partial x \partial y}\Big|_{(x,-t^*x)}\Big\} < 0,$$

in which case $x^* > x_0$, $y^* < y_0$ if $b > -a$, and $x^* > x_0$, $y^* > y_0$ if $b < -a$.

2.

$$\frac{\partial F}{\partial x}\Big|_{(x_0,y_0)} > 0, \qquad \inf_{x\le 0}\left\{a+(1+t^*)\frac{\partial^2 F}{\partial x\partial y}\Big|_{(x,-t^*x)}\right\}>0,$$
$$or \quad \frac{\partial F}{\partial x}\Big|_{(x_0,y_0)} < 0, \qquad \sup_{x\le 0}\left\{a+(1+t^*)\frac{\partial^2 F}{\partial x\partial y}\Big|_{(x,-t^*x)}\right\}<0,$$

in which case $x^* < x_0$, $y^* > y_0$ if $b > -a$, and $x^* < x_0$, $y^* < y_0$ if $b < -a$.

Proof of Lemma A.2.

Without loss of generality, we assume $(x_0, y_0) = (0, 0)$. Denote

$$\phi(x,y) \equiv \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} + b \qquad \text{and} \qquad \psi(x,y) \equiv \frac{\partial^2 F}{\partial x \partial y}.$$

Let $t > 0$ and define $f(x)$ and $g(x)$ as the following:

$$f(x) = \frac{\partial F}{\partial x}\Big|_{(x,-tx)}, \quad g(x) = \frac{\partial F}{\partial y}\Big|_{(x,-tx)}, \quad \frac{df}{dx} = \phi(x,-tx) - t\psi(x,-tx), \quad \frac{dg}{dx} = \psi(x,-tx) - t\big(\phi(x,-tx)-b\big).$$

We solve for t^* and h^* such that $f(h^*) = g(h^*) = 0$. The system of equations is the following

$$\begin{aligned} 0 = f(h) &= f(0) + \int_0^h \phi(x,-tx)dx - t\int_0^h \psi(x,-tx)dx \\ 0 = g(h) &= g(0) + \int_0^h \psi(x,-tx)dx - t\int_0^h \big(\phi(x,-tx)-b\big)dx \\ &= g(0) + \int_0^h \psi(x,-tx)dx - t\int_0^h \phi(x,-tx)dx + tbh \end{aligned}$$

Adding the two equations above and noting that $f(0) + g(0) = 0$, we obtain

$$\begin{aligned} tbh + (1-t)\left(\int_0^h \phi(x,-tx)dx + \int_0^h \psi(x,-tx)dx\right) &= 0 \iff \\ tbh + (1-t)\int_0^h \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x\partial y}\right)dx &= 0 \iff \\ tbh - (1-t)ah &= 0 \iff \\ t(b+a) &= a \end{aligned}$$

Let $t^* = \frac{a}{a+b}$. It remains to find h^* that solves the following equation,

$$\begin{aligned} &\int_0^{h^*}\Big(t^*\psi(x,-t^*x) - \phi(x,-t^*x)\Big)dx = f(0) \\ \iff &\int_0^{h^*}\left(a + (1+t^*)\frac{\partial^2 F}{\partial x\partial y}\Big|_{(x,-t^*x)}\right)dx = f(0). \end{aligned}$$

The existence of $h^* > 0$ is guaranteed if either

$$\inf_{x\ge 0}\left\{a+(1+t^*)\frac{\partial^2 F}{\partial x\partial y}\Big|_{(x,-t^*x)}\right\}>0, \quad f(0)>0$$

or

$$\sup_{x\geq 0}\left\{a+(1+t^*)\frac{\partial^2 F}{\partial x\partial y}\bigg|_{(x,-t^*x)}\right\}<0, \quad f(0)<0$$

in which case $x^* > x_0$. Clearly, $y^* < y_0$ if $a+b>0$ and $y^* > y_0$ if $a+b<0$. The other case can be similarly proved. ■.

Proof of Proposition 5.

We first prove the profit function $\Pi(p_H,p_L)=\pi^0(p_H,p_L)+\pi^*(p_H,p_L)$ is strictly concave. We do not require $\pi^0(p_H,p_L)$ be strictly concave because we only require (p_h^*,p_l^*) satisfy the first-order condition later in the proof.

Because $\pi^0(p_H,p_L)=(p_H-c_H)(1-w_0)+(p_L-c_L)(w_0-\underline{w})$, taking derivative with respect to p_H and p_L yields

$$\begin{aligned}\frac{\partial\pi^0(p_H,p_L)}{\partial p_H} &= 1-w_0-(p_H-p_L-c)\frac{\partial w_0}{\partial p_H}\\ \frac{\partial\pi^0(p_H,p_L)}{\partial p_L} &= w_0-\underline{w}-(p_L-c_L)\frac{\partial \underline{w}}{\partial p_L}-(p_H-p_L-c)\frac{\partial w_0}{\partial p_L}\end{aligned}$$

Using Equation (A.2), we have

$$\frac{\partial\pi^0(p_H,p_L)}{\partial p_H}+\frac{\partial\pi^0(p_H,p_L)}{\partial p_L}=1-(p_H-p_L-c)-p_L\frac{\partial \underline{w}}{\partial p_L}+c_L\frac{\partial \underline{w}}{\partial p_L}-\underline{w}$$

Taking derivative of the above equation with respect to p_H and p_L yields

$$\begin{aligned}\frac{\partial^2\pi^0(p_H,p_L)}{\partial p_H^2}+\frac{\partial^2\pi^0(p_H,p_L)}{\partial p_H\partial p_L} &= -1\\ \frac{\partial^2\pi^0(p_H,p_L)}{\partial p_L^2}+\frac{\partial^2\pi^0(p_H,p_L)}{\partial p_H\partial p_L} &= 1-2\frac{\partial \underline{w}}{\partial p_L}-(p_L-c_L)\frac{\partial^2 \underline{w}}{\partial p_L^2}\end{aligned}$$

By Proposition 4,

$$\frac{\partial\pi^*(p_H,p_L)}{\partial p_H}+\frac{\partial\pi^*(p_H,p_L)}{\partial p_L}=0.$$

Taking partial derivative of the above equality with respect to p_H and p_L yields

$$\frac{\partial^2\pi^*(p_H,p_L)}{\partial p_H^2}+\frac{\partial^2\pi^*(p_H,p_L)}{\partial p_H\partial p_L}=0, \quad \text{and} \quad \frac{\partial^2\pi^*(p_H,p_L)}{\partial p_H\partial p_L}+\frac{\partial^2\pi^*(p_H,p_L)}{\partial p_L^2}=0.$$

Let

$$A=\frac{\partial^2\pi^0(p_H,p_L)}{\partial p_H^2}, \qquad B=\frac{\partial^2\pi^*(p_H,p_L)}{\partial p_H^2}.$$

The Hessian matrix of $\Pi(p_H,p_L)=\pi^0(p_H,p_L)+\pi^*(p_H,p_L)$ is

$$\begin{aligned} &\begin{bmatrix} A & -1-A \\ -1-A & A+2-2\frac{\partial \underline{w}}{\partial p_L}-(p_L-c_L)\frac{\partial^2 \underline{w}}{\partial p_L^2}\end{bmatrix}+\begin{bmatrix} B & -B \\ -B & B\end{bmatrix}\\ =&\begin{bmatrix} A+B & -1-A-B \\ -1-A-B & A+B+2-2\frac{\partial \underline{w}}{\partial p_L}-(p_L-c_L)\frac{\partial^2 \underline{w}}{\partial p_L^2}\end{bmatrix}.\end{aligned}$$

$\Pi(p_H,p_L)$ is strictly concave if and only if its Hessian matrix is negative definite, i.e.,

$$\begin{aligned}
&\iff \quad A+B<0 \quad \text{and} \quad (A+B)^2+\big(2-2\frac{\partial \underline{w}}{\partial p_L}-(p_L-c_L)\frac{\partial^2 \underline{w}}{\partial p_L^2}\big)(A+B)-(A+B+1)^2>0\\
&\iff \quad A+B<0 \quad \text{and} \quad (A+B)\left(2\frac{\partial \underline{w}}{\partial p_L}+(p_L-c_L)\frac{\partial^2 \underline{w}}{\partial p_L^2}\right)<-1\\
&\iff \quad A+B<-\frac{1}{2\frac{\partial \underline{w}}{\partial p_L}+(p_L-c_L)\frac{\partial^2 \underline{w}}{\partial p_L^2}}<0
\end{aligned}$$

Using $U(\underline{w},q_0)=U(\underline{w}-p_L,q_L)$, we obtain

$$\frac{\partial \underline{w}}{\partial p_L}=\frac{U_1(\underline{w}-p_L,q_L)}{U_1(\underline{w}-p_L,q_L)-U_1(\underline{w},q_0)}>1$$

Because we have assumed $\underline{w}$ is linear in p_L, $\Pi(p_H,p_L)$ is strictly concave if

$$\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H^2}=A+B<-\frac{1}{2}.$$

We now show the above condition is equivalent to condition (7).

$$\begin{aligned}
\frac{\partial^2\pi^0(p_H,p_L)}{\partial p_H^2} &= -2\frac{\partial w_0}{\partial p_H}-(p_H-p_L-c)\frac{\partial^2 w_0}{\partial p_H^2}\\
\frac{\partial\pi^*(p_H,p_L)}{\partial p_H} &= w_0-w_H+\big(p^*-\lambda^*p_H-(1-\lambda^*)p_L\big)\frac{\partial w_H}{\partial p_H}+(p_H-p_L-c)\Big(\frac{\partial w_0}{\partial p_H}-(1-\lambda^*)\frac{\partial w_H}{\partial p_H}\Big)\\
&= w_0-w_H+\frac{\partial w_H}{\partial p_H}\big(p^*-p_H+c(1-\lambda^*)\big)+(p_H-p_L-c)\frac{\partial w_0}{\partial p_H}\\
\frac{\partial^2\pi^*(p_H,p_L)}{\partial p_H^2} &= 2\frac{\partial w_0}{\partial p_H}-\frac{\partial^* w_H}{\partial p_H}+(p_H-p_L-c)\frac{\partial^2 w_0}{\partial p_H^2}+\frac{\partial w_H}{\partial p_H}\Big(\frac{\partial p^*}{\partial p_H}-1-c\frac{\partial\lambda^*}{\partial p_H}\Big)+\big(p^*-p_H+c(1-\lambda^*)\big)\frac{\partial^*}{\partial p_H}\frac{\partial w_H}{\partial p_H}\\
\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H^2} &= \frac{\partial w_H}{\partial p_H}\left(\frac{\partial p^*}{\partial p_H}-c\frac{\partial\lambda^*}{\partial p_H}-2\right)+\big(p-p_H+(1-\lambda^*)c\big)\frac{\partial^*}{\partial p_H}\Big(\frac{\partial w_H}{\partial p_H}\Big),
\end{aligned}$$

$\Pi(p_H,p_L)$ is strictly concave if

$$\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H^2}<-\frac{1}{2} \iff \frac{\partial w_H}{\partial p_H}\Big(2-\frac{\partial p^*}{\partial p_H}+c\frac{\partial\lambda^*}{\partial p_H}\Big)>\frac{1}{2}+\big(p^*-p_H+(1-\lambda^*)c\big)\frac{\partial^*}{\partial p_H}\Big(\frac{\partial w_H}{\partial p_H}\Big),$$

which is condition (7).

We now apply Lemma A.2 to prove $p_H^*-p_h^*\ge p_l^*-p_L^*>0$, with $F(x,y)=\Pi(p_H,p_L)$ and $(x_0,y_0)=(p_h^*,p_l^*)$.

First, we need to check the assumptions of Lemma A.2. $\Pi(p_H,p_L)$ satisfies the two partial differential equations of Lemma A.2 with

$$a=1,\quad b=2\left(\frac{\partial \underline{w}}{\partial p_L}-1\right)>0$$

$$\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H^2}+\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H\partial p_L}=\overbrace{\frac{\partial^2\pi^0(p_H,p_L)}{\partial p_H^2}+\frac{\partial^2\pi^0(p_H,p_L)}{\partial p_H\partial p_L}}^{-1}+\overbrace{\frac{\partial^2\pi^*(p_H,p_L)}{\partial p_H^2}+\frac{\partial^2\pi^*(p_H,p_L)}{\partial p_H\partial p_L}}^{0}=-1$$

$$\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H^2}-\frac{\partial^2\Pi(p_H,p_L)}{\partial p_L^2}=2\left(\frac{\partial \underline{w}}{\partial p_L}-1\right)+(p_L-c_L)\frac{\partial^2\underline{w}}{\partial p_L^2}=2\left(\frac{\partial \underline{w}}{\partial p_L}-1\right)\equiv b.$$

Because $a=1$, $b>0$, and $A+B<-\frac{1}{2}$, we have

$$a+(1+t^*)\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H\partial p_L}=1+\frac{b+2}{b+1}\frac{\partial^2\Pi(p_H,p_L)}{\partial p_H\partial p_L}=1-\frac{b+2}{b+1}(1+A+B)>1-\frac{b+2}{2(b+1)}=\frac{b}{2(b+1)}>0.$$

The condition for $(x_0,y_0)=(p_h^*,p_l^*)$ follows from Proposition 4 and the fact that (p_h^*,p_l^*) maximizes $\pi^0(p_H,p_L)$:

$$\begin{aligned}&\left.\frac{\partial\Pi(p_H,p_L)}{\partial p_H}\right|_{(p_h^*,p_l^*)}+\left.\frac{\partial\Pi(p_H,p_L)}{\partial p_L}\right|_{(p_h^*,p_l^*)}\\ =&\left.\frac{\partial\pi^0(p_H,p_L)}{\partial p_H}\right|_{(p_h^*,p_l^*)}+\left.\frac{\partial\pi^0(p_H,p_L)}{\partial p_L}\right|_{(p_h^*,p_l^*)}+\left.\frac{\partial\pi^*(p_H,p_L)}{\partial p_H}\right|_{(p_h^*,p_l^*)}+\left.\frac{\partial\pi^*(p_H,p_L)}{\partial p_L}\right|_{(p_h^*,p_l^*)}\\ =&\ 0+0+\left.\left(\frac{\partial\pi^*(p_H,p_L)}{\partial p_H}+\frac{\partial\pi^*(p_H,p_L)}{\partial p_L}\right)\right|_{(p_h^*,p_l^*)}=0.\end{aligned}$$

Finally, we show $\partial\pi^*(p_H,p_L)/\partial p_H>0$ at (p_h^*,p_l^*) if and only if (8) holds. We rewrite (A.12) and (A.13), the first-order conditions for the optimal solution (p^*,λ^*), as

$$w_H-w_L+\frac{\partial w_H}{\partial p}(p-p_H+(1-\lambda)c)-\frac{\partial w_L}{\partial p}(p-p_L-\lambda c)=0 \tag{A.16}$$

$$\frac{\partial(w_H-w_L)}{\partial\lambda}p=\frac{\partial w_H}{\partial\lambda}p_H-\frac{\partial w_L}{\partial\lambda}p_L+c\Big(w_H-w_L-(1-\lambda)\frac{\partial w_H}{\partial\lambda}-\lambda\frac{\partial w_L}{\partial\lambda}\Big) \tag{A.17}$$

Because

$$\begin{aligned}&\partial\pi^*(p_H,p_L)/\partial p_H>0\\ \iff\ & w_0-w_L+\big(p-\lambda p_H-(1-\lambda)p_L\big)\frac{\partial w_H}{\partial p_H}+(p_H-p_L-c)\Big(\frac{\partial w_0}{\partial p_H}-(1-\lambda)\frac{\partial w_H}{\partial p_H}\Big)>w_H-w_L,\end{aligned}$$

by (A.16), this is equivalent to

$$\begin{aligned}&w_0-w_L+\big(p-\lambda p_H-(1-\lambda)p_L\big)\frac{\partial w_H}{\partial p_H}+(p_H-p_L-c)\Big(\frac{\partial w_0}{\partial p_H}-(1-\lambda)\frac{\partial w_H}{\partial p_H}\Big)\\ >&\ \frac{\partial w_L}{\partial p}(p-p_L-\lambda c)-\frac{\partial w_H}{\partial p}(p-p_H+(1-\lambda)c)\\ =&\ (p_H-p_L-c)\Big(\lambda\frac{\partial w_L}{\partial p}+(1-\lambda)\frac{\partial w_H}{\partial p}\Big)-\big(p-\lambda p_H-(1-\lambda)p_L\big)\Big(\frac{\partial w_H}{\partial p}-\frac{\partial w_L}{\partial p}\Big).\end{aligned}$$

Using (A.11), the above implies to

$$\begin{aligned}&\big(p-\lambda p_H-(1-\lambda)p_L\big)\frac{\partial w_L}{\partial p_L}+w_0-w_L>(p_H-p_L-c)\Big(\frac{\partial w_0}{\partial p_L}-\lambda\frac{\partial w_L}{\partial p_L}\Big)\\ \iff\ &(p-p_L-\lambda c)\frac{\partial w_L}{\partial p_L}-w_L>(p_H-p_L-c)\frac{\partial w_0}{\partial p_L}-w_0.\end{aligned}$$

At $(p_H, p_L) = (p_h^*, p_l^*)$, the prices of the component goods satisfy the first-order conditions of the following optimization problem

$$\max_{p_H, p_L} \pi^0(p_H, p_L) = (p_H - c_H)(1 - w_0) + (p_L - c_L)(w_0 - \underline{w}).$$

In particular, the first-order condition with respect to p_L is

$$-(p_H - c_H)\frac{\partial w_0}{\partial p_L} + w_0 - \underline{w} + (p_L - c_L)\Big(\frac{\partial w_0}{\partial p_L} - \frac{\partial \underline{w}}{\partial p_L}\Big) = 0 \iff (p_H - p_L - c)\frac{\partial w_0}{\partial p_L} = w_0 - \underline{w} - (p_L - c_L)\frac{\partial \underline{w}}{\partial p_L}$$

Hence, $\partial \pi^*(p_H, p_L)/\partial p_H > 0$ if and only if

$$(p - p_L - \lambda c)\frac{\partial w_L}{\partial p_L} > w_L - \underline{w} - (p_L - c_L)\frac{\partial \underline{w}}{\partial p_L}$$

which is condition (8).

The conclusion now follows from Lemma A.2 and the fact that $\Pi(p_H, p_L)$ is strictly concave which ensures the sufficiency of the first-order condition. Moreover, from the proof of Lemma A.2, we have

$$t^* = \frac{1}{b+1} \leq 1 \Longrightarrow p_H^* - p_h^* \geq p_l^* - p_L^*.$$

■.

B. Canonical Utility

Proof of Lemma 1.

Taking derivative of $\pi(p,\lambda)$ with respect to p and setting it zero yields

$$\lambda p_H\gamma + (1-\lambda)p_L(\gamma-1) - p(\gamma-1+\lambda) - (\gamma-1+\lambda)\big(p-\lambda p_H-(1-\lambda)p_L\big) = 0,$$

$$\Longrightarrow p = \frac{\lambda(2\gamma-1+\lambda)p_H + (1-\lambda)(2\gamma-2+\lambda)p_L}{2(\gamma-1+\lambda)} \tag{B.1}$$

To derive $\partial\pi(p,\lambda)/\partial\lambda$, we first collect the terms in $\pi(p,\lambda)$ involving λ, which yields

$$\frac{1}{1-\lambda}\left(pp_H\gamma + (p-p_H)\gamma p_H - p^2\gamma\right) + \frac{1}{\lambda}\left(p^2 + pp_L(\gamma-1) + (p-p_L)(\gamma-1)p_L - p^2\gamma\right).$$

Taking derivative of the above expression with respect to λ yields

$$\frac{(\gamma-1)(p-p_L)^2}{\lambda^2} - \frac{\gamma(p_H-p)^2}{(1-\lambda)^2}.$$

Setting the above expression to zero, we obtain

$$\left(\frac{1-\lambda}{\lambda}\right)^2 = \frac{\gamma}{\gamma-1}\cdot\frac{(p_H-p)^2}{(p_L-p)^2} \tag{B.2}$$

Using (B.1), we have

$$p_H - p = \frac{(1-\lambda)(2\gamma-2+\lambda)(p_H-p_L)}{2(\gamma-1+\lambda)}, \qquad p - p_L = \frac{\lambda(2\gamma-1+\lambda)(p_H-p_L)}{2(\gamma-1+\lambda)}.$$

Substitute the expressions above into (B.2), we obtain

$$\frac{\gamma-1}{\gamma} = \left(1+\frac{1}{1-2\gamma-\lambda}\right)^2,$$

from which we can easily solve for its unique non-negative solution $\lambda^* = \sqrt{\gamma(\gamma-1)} - \gamma + 1$. Substituting λ^* into the second equation in (B.2), we obtain

$$\left(\frac{p_H-p}{p_L-p}\right)^2 = \frac{\gamma-1}{\gamma}\left(\frac{1-\lambda}{\lambda}\right)^2 = 1,$$

from which we have $p^* = (p_H+p_L)/2$. Substitute p^* into $\pi(p,\lambda)$ to obtain

$$\begin{aligned}
\pi^*(p_H,p_L) &= \big(\frac{1}{2}-\lambda^*\big)(p_H-p_L)\frac{\lambda^*\gamma p_H + (1-\lambda^*)(\gamma-1)p_L - \frac{\gamma-1+\lambda^*}{2}(p_H+p_L)}{\lambda^*(1-\lambda^*)} \\
&= \big(\frac{1}{2}-\lambda^*\big)(p_H-p_L)\frac{(2\lambda^*\gamma-\gamma+1-\lambda^*)p_H - \big(\gamma-1+\lambda^*-2(1-\lambda^*)(\gamma-1)\big)p_L}{2\lambda^*(1-\lambda^*)} \\
&= \left(\frac{1}{2}-\lambda^*\right)\frac{2\lambda^*\gamma-\gamma+1-\lambda^*}{2\lambda^*(1-\lambda^*)}\big(p_H-p_L\big)^2.
\end{aligned}$$

■.

Proof of Proposition 6 .

For ease of exposition, we simplify the notation of (p^*, λ^*) by (p, λ). The total profit is

$$\begin{aligned}\Pi(p_H, p_L) &= \big(p-\lambda c_H-(1-\lambda)c_L\big)(w_H-w_L)+(p_H-c_H)(1-w_H)+(p_L-c_L)(w_L-\underline{w})\\ &= \Big(p-\lambda c_H-(1-\lambda)c_L\Big)\frac{\lambda p_H\gamma+(1-\lambda)p_L(\gamma-1)-p(\gamma-1+\lambda)}{\lambda(1-\lambda)}\\ &\quad +(p_H-c_H)(1-w_H)+(p_L-c_L)(w_L-\tau p_L)\end{aligned}$$

where $\underline{w}=\tau p_L$, $\tau\equiv q_L/(q_L-q_0)>1$, $p=\frac{p_H+p_L}{2}$, and $\lambda=\sqrt{\gamma(\gamma-1)}-\gamma+1$ by Lemma 1.

The first-order conditions with respect to p_H and p_L are given below:

$$\begin{aligned}\frac{\partial\Pi}{\partial p_H} &= \frac{\lambda p_H\gamma+(1-\lambda)p_L(\gamma-1)-p(\gamma-1+\lambda)}{2\lambda(1-\lambda)}+\big(p-\lambda c-c_L\big)\frac{\lambda\gamma-\frac{1}{2}(\gamma-1+\lambda)}{\lambda(1-\lambda)}+\\ &\quad\left(1-\frac{p_H}{1-\lambda}\gamma+\frac{p}{1-\lambda}(\gamma-1+\lambda)\right)+(p_H-c_H)\left(\frac{\gamma-1+\lambda}{2(1-\lambda)}-\frac{\gamma}{1-\lambda}\right)+(p_L-c_L)\frac{\gamma-1+\lambda}{2\lambda}=0\\ \frac{\partial\Pi}{\partial p_L} &= \frac{\lambda p_H\gamma+(1-\lambda)p_L(\gamma-1)-p(\gamma-1+\lambda)}{2\lambda(1-\lambda)}+\big(p-\lambda c-c_L\big)\frac{(1-\lambda)(\gamma-1)-\frac{1}{2}(\gamma-1+\lambda)}{\lambda(1-\lambda)}+\\ &\quad(p_H-c_H)\frac{\gamma-1+\lambda}{2(1-\lambda)}+\frac{\gamma-1+\lambda}{2\lambda}p_H-\frac{\gamma-1+\lambda(2\tau-1)}{\lambda}p_L+\frac{\gamma-1+\lambda(2\tau-1)}{2\lambda}c_L=0\end{aligned}$$

Collecting terms for p_H, p_L, and p, we obtain

$$\begin{aligned}(\lambda^2-2\lambda\gamma-\lambda)p_H+(1-\lambda)(\lambda+2\gamma-2)p_L+2p(\lambda^2+2\lambda\gamma-2\lambda-\gamma+1)+2\lambda(1-\lambda)\big(1+\gamma c\big) &= 0\\ (\lambda+\gamma+\lambda\gamma-1)p_H+(1-\lambda)\big(1-\gamma-2\lambda(2\tau-1)\big)p_L-2\lambda\gamma p-2\lambda(1-\lambda)\big((\gamma-1)c_H-(\gamma-1+\tau)c_L\big) &= 0\end{aligned}$$

Using the closed-form solution of (p, λ), we have the following system of linear equations:

$$\begin{aligned}\frac{(\gamma-1+\lambda)+2\lambda(1-\lambda)}{2\lambda(1-\lambda)}p_H-\frac{\gamma-1+\lambda}{2\lambda(1-\lambda)}p_L &= 1+\gamma c\\ \frac{\gamma-1+\lambda}{2\lambda(1-\lambda)}p_H-\frac{(\gamma-1+\lambda)+2\lambda(1-\lambda)(2\tau-1)}{2\lambda(1-\lambda)}p_L &= (\gamma-1)c_H-(\gamma-1+\tau)c_L\end{aligned}$$

Denoting

$$\kappa\equiv\frac{\gamma-1+\lambda}{2\lambda(1-\lambda)}=\frac{\sqrt{\gamma(\gamma-1)}}{2(\gamma-\sqrt{\gamma(\gamma-1)}(\sqrt{\gamma(\gamma-1)}-\gamma+1)}=\frac{1}{2}(\sqrt{\gamma}-\sqrt{\gamma-1})^{-2}$$

and noticing that

$$\frac{(\gamma-1+\lambda)+2\lambda(1-\lambda)}{2\lambda(1-\lambda)}=\kappa+1,$$

we can rewrite the system of equations as

$$\begin{cases}(\kappa+1)p_H-\kappa p_L = 1+\gamma c\\ \kappa p_H-(\kappa+1+2\tau-2)p_L = (\gamma-1)c_H-(\gamma-1+\tau)c_L\end{cases}$$

from which we can explicitly solve for (p_H^*, p_L^*) as

$$p_H^*=\frac{\kappa+2\tau-1+\big((2\tau-1)\gamma+\kappa\big)c_H+\big(\kappa(\tau-1)-\gamma(2\tau-1)\big)c_L}{2\tau(\kappa+1)-1}$$

$$p_L^* = \frac{\kappa + (\kappa - \gamma + 1)c_H + \big((\tau-1)(\kappa+1)+\gamma\big)c_L}{2\tau(\kappa+1)-1}$$
$$p^* = \frac{p_H^* + p_L^*}{2} = \frac{2\kappa + 2\tau - 1 + \big(2\kappa + 1 + 2\gamma(\tau-1)\big)c_H + (\tau-1)(2\kappa - 2\gamma + 1)c_L}{4\tau(\kappa+1)-2}.$$

Finally, using

$$p_l^* = \frac{2\gamma - 1 + \gamma c_H + \gamma(2\tau-1)c_L}{4\gamma(\gamma-1+\tau)-(1-2\gamma)^2},$$

we calculate the relative increase[14] of market coverage after introducing probabilistic selling. The market coverage is $1 - \underline{w}(p_L^*) = 1 - p_L^*\tau$ with probabilistic selling and $1 - \underline{w}(p_l^*) = 1 - p_l^*\tau$ without probabilistic selling, hence the relative increase of market coverage is

$$\begin{aligned}
\frac{p_l^* - p_L^*}{1 - p_l^*\tau}\tau &= \frac{\frac{2\gamma-1+\gamma c_H+\gamma(2\tau-1)c_L}{4\gamma\tau-1} - \frac{\kappa+(\kappa-\gamma+1)c_H+((\tau-1)(\kappa+1)+\gamma)c_L}{2\tau(\kappa+1)-1}}{1 - \frac{2\gamma-1+\gamma c_H+\gamma(2\tau-1)c_L}{4\gamma\tau-1}\tau}\tau \\
&= \frac{2\gamma - 1 + \gamma c_H + \gamma(2\tau-1)c_L - \frac{\kappa(4\gamma\tau-1)+(4\gamma\tau-1)(\kappa-\gamma+1)c_H+(4\gamma\tau-1)((\tau-1)(\kappa+1)+\gamma)c_L}{2\tau(\kappa+1)-1}}{2\gamma\tau - 1 + \tau - \gamma\tau c_H - \gamma\tau(2\tau-1)c_L}\tau \\
&= \frac{(2\gamma-1) - \frac{\kappa(4\gamma\tau-1)}{2\tau(\kappa+1)-1} + \left(\gamma - \frac{(4\gamma\tau-1)(\kappa-\gamma+1)}{2\tau(\kappa+1)-1}\right)c_H + \left(\gamma(2\tau-1) - \frac{(4\gamma\tau-1)((\tau-1)(\kappa+1)+\gamma)}{2\tau(\kappa+1)-1}\right)c_L}{2\gamma\tau - 1 + \tau - \gamma\tau c_H - \gamma\tau(2\tau-1)c_L}\tau \\
&= \frac{\frac{(2\gamma-\kappa-1)(2\tau-1)}{2\tau(\kappa+1)-1} - \frac{(2\gamma-\kappa-1)(1-2\gamma\tau)}{2\tau(\kappa+1)-1}c_H - \frac{(2\gamma-\kappa-1)(2\gamma\tau+\tau-1)}{2\tau(\kappa+1)-1}c_L}{2\gamma\tau - 1 + \tau - \gamma\tau c_H - \gamma\tau(2\tau-1)c_L}\tau \\
&= \frac{(2\gamma-\kappa-1)(2\tau-1) + (\kappa+1-2\gamma)(1-2\gamma\tau)c_H + (\kappa+1-2\gamma)(2\gamma\tau+\tau-1)c_L}{\big(2\tau(\kappa+1)-1\big)\big(2\gamma\tau-1+\tau-\gamma\tau c_H - \gamma\tau(2\tau-1)c_L\big)}\tau \\
&= \frac{2\gamma-\kappa-1}{2\tau(\kappa+1)-1} \cdot \frac{2\tau - 1 - (1-2\gamma\tau)c_H - (2\gamma\tau+\tau-1)c_L}{2\gamma\tau - 1 + \tau - \gamma\tau c_H - \gamma\tau(2\tau-1)c_L}.
\end{aligned}$$

■.

Proof of Lemma 2.

First, we extend the index set from $\{1, \cdots, n\}$ to $\{0, 1, \cdots, n, n+1\}$ by assigning index $i = 0$ to the high-quality good and $i = n+1$ to the low-quality good. Clearly, for the menu of probabilistic goods to be optimal, $\forall i \in \{1, \cdots, n\}$, the i-th probabilistic good must be optimal when it is considered as the *single* probabilistic good constructed from the $(i-1)$-th good and the $(i+1)$-th good. Otherwise, one can always adjust the design of the i-th probabilistic good to improve the profit due to the selling of the i-th probabilistic good, thereby the total profit as well.

The quality of the i-th probabilistic good, denoted by q_i, can be written as $q_i = \lambda_i q_H + (1-\lambda_i)q_L$. Applying Lemma 1, we have, $\forall i \in \{1, \cdots, n\}$,

$$p_i = \frac{p_{i-1} + p_{i+1}}{2} \text{ and } q_i = \sqrt{q_{i-1}q_{i+1}},$$

[14] The relative increase of x_2 over x_1 is defined as $(x_2 - x_1)/x_1$.

or equivalently,

$$p_{i+1}-p_i=p_i-p_{i-1} \text{ and } \frac{q_{i+1}}{q_i}=\frac{q_i}{q_{i-1}}.$$

In other words, the sequence of optimal prices forms a arithmetic sequence while the sequence of qualities forms a geometric sequence. Therefore, we immediately have

$$p_i=\frac{p_H-p_L}{n+1}(n-i+1)+p_L=\frac{n+1-i}{n+1}p_H+\frac{i}{n+1}p_L, \text{ and } q_i=r^{\frac{i}{n+1}}q_H.$$

Again, using the fact that $q_i=\lambda_i q_H+(1-\lambda_i)q_L$, we have

$$\lambda_i=\frac{q_i-q_L}{q_H-q_L}=\frac{r^{\frac{i}{n+1}}-r}{1-r}.$$

We think of the optimal n probabilistic goods on the menu as a sequence of goods introduced one by one according to the ascending order of the index i. The first probabilistic good q_1 is introduced as a mix combining q_H and q_L. When introducing q_i, it is obtained as a mix combining q_{i-1} and q_L, i.e., $\{\lambda_i', q_{i-1}; (1-\lambda_i'), q_L\}$. In this way, we can calculate the incremental profit from introducing q_i to the menu according to π in problem (9). According to the definition of optimal offering probability λ_i, we have $\lambda_i=\lambda_i'\cdot\lambda_{i-1}$. Thus, the optimal profit from selling n probabilistic good $\pi(n)$ based on the optimal pricing and probability is given by

$$\pi(n)=\sum_{i=1}^{n}\pi_i(\lambda_i',p_i) \tag{B.3}$$

with $\lambda_i'=\frac{\lambda_i}{\lambda_{i-1}}$, $\gamma_i=\frac{q_{i-1}}{q_{i-1}-q_L}$, and $q_i=\lambda_i q_H+(1-\lambda_i)q_L$.

For any $i\in\{1,2,...,n\}$, the profit increase is given by

$$\pi_i(\lambda_i',p_i)=(p_i-\lambda_i'(p_{i-1}-p_L)-p_L)\frac{\lambda_i' p_{i-1}\gamma_i+(1-\lambda_i')p_L(\gamma_i-1)-p_i(\gamma_i-1+\lambda_i')}{\lambda_i'(1-\lambda_i')}$$

From the main text, we have

$$\begin{aligned}
p_i=&\frac{n+1-i}{n+1}\Delta p+p_L, \Delta p=p_H-p_L, \lambda_i=\frac{r^{\frac{i}{n+1}}-r}{1-r}, \lambda_i'=\frac{\lambda_i}{\lambda_{i-1}}=\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\\
\gamma_i=&\frac{q_{i-1}}{q_{i-1}-q_L}=\frac{\lambda_{i-1}(q_H-q_L)+q_L}{\lambda_{i-1}(q_H-q_L)}=1+\frac{q_L}{q_H-q_L}\frac{1}{\lambda_{i-1}}
\end{aligned}$$

Substituting these expressions into π_i, we have

$$\begin{aligned}
&\pi_i(p_i,\lambda_i')=\\
&\left(\frac{n+1-i}{n+1}\Delta p-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\Delta p\right)\\
&\left(\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(1+\frac{q_L}{q_H-q_L}\frac{1-r}{r^{\frac{i-1}{n+1}}-r}\right)}{1-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}}+\frac{p_L\frac{q_L}{q_H-q_L}\frac{1-r}{r^{\frac{i-1}{n+1}}-r}}{\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}}-\frac{\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{q_L}{q_H-q_L}\frac{1-r}{r^{\frac{i-1}{n+1}}-r}+\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\right)}{\left(\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\right)\left(1-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\right)}\right)
\end{aligned}$$

$$=\Delta p\left(\frac{n+1-i}{n+1}-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\right)$$
$$\left(\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(r^{\frac{i-1}{n+1}}-r+\frac{q_L}{q_H-q_L}(1-r)\right)}{r^{\frac{i-1}{n+1}}-r^{\frac{i}{n+1}}}+\frac{p_L\frac{q_L}{q_H-q_L}(1-r)}{r^{\frac{i}{n+1}}-r}-\frac{\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{q_L}{q_H-q_L}(1-r)+r^{\frac{i}{n+1}}-r\right)}{\left(r^{\frac{i}{n+1}}-r\right)\left(1-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\right)}\right)$$

Notice that $\frac{q_L}{q_H-q_L}(1-r)=r$, we have

$$\pi_i(p_i,\lambda_i')=$$
$$=\Delta p\left(\frac{n+1-i}{n+1}-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\right)\left(\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(r^{\frac{i-1}{n+1}}\right)}{r^{\frac{i-1}{n+1}}-r^{\frac{i}{n+1}}}+\frac{p_Lr}{r^{\frac{i}{n+1}}-r}-\frac{\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(r^{\frac{i}{n+1}}\right)}{\left(r^{\frac{i}{n+1}}-r\right)\left(1-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\right)}\right)$$
$$=\Delta p\left(\frac{n+1-i}{n+1}-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\right)\left(\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)}{1-r^{\frac{1}{n+1}}}+\frac{p_Lr}{r^{\frac{i}{n+1}}-r}-\frac{\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)r^{\frac{i}{n+1}}(r^{\frac{i-1}{n+1}}-r)}{\left(r^{\frac{i}{n+1}}-r\right)\left(r^{\frac{i-1}{n+1}}-r^{\frac{i}{n+1}}\right)}\right)$$

After expanding, we have

$$\pi_i(p_i,\lambda_i')=\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)}{1-r^{\frac{1}{n+1}}}\frac{n+1-i}{n+1}\Delta p+\frac{p_Lr}{r^{\frac{i}{n+1}}-r}\frac{n+1-i}{n+1}\Delta p-\frac{\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)r^{\frac{i}{n+1}}(r^{\frac{i-1}{n+1}}-r)}{\left(r^{\frac{i}{n+1}}-r\right)\left(r^{\frac{i-1}{n+1}}-r^{\frac{i}{n+1}}\right)}\frac{n+1-i}{n+1}\Delta p$$
$$-\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)}{1-r^{\frac{1}{n+1}}}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\Delta p-\frac{p_Lr}{r^{\frac{i}{n+1}}-r}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\Delta p$$
$$+\frac{\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)r^{\frac{i}{n+1}}(r^{\frac{i-1}{n+1}}-r)}{\left(r^{\frac{i}{n+1}}-r\right)\left(r^{\frac{i-1}{n+1}}-r^{\frac{i}{n+1}}\right)}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\Delta p$$

$$=\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)}{1-r^{\frac{1}{n+1}}}\frac{n+1-i}{n+1}\Delta p+\frac{p_Lr}{r^{\frac{i}{n+1}}-r}\frac{n+1-i}{n+1}\Delta p-\frac{\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)r^{\frac{i}{n+1}}(r^{\frac{i-1}{n+1}}-r)}{\left(r^{\frac{i}{n+1}}-r\right)\left(r^{\frac{i-1}{n+1}}-r^{\frac{i}{n+1}}\right)}\frac{n+1-i}{n+1}\Delta p$$
$$-\frac{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)}{1-r^{\frac{1}{n+1}}}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\Delta p-\frac{p_Lr}{r^{\frac{i}{n+1}}-r}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\frac{n+2-i}{n+1}\Delta p$$
$$+\frac{r^{\frac{i}{n+1}}}{r^{\frac{i-1}{n+1}}-r^{\frac{i}{n+1}}}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\frac{n+2-i}{n+1}\Delta p$$

$$=\frac{1}{1-r^{\frac{1}{n+1}}}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p+\frac{r}{r^{\frac{i}{n+1}}-r}\left(\frac{n+1-i}{n+1}\right)p_L\Delta p$$
$$-\frac{1}{r^{\frac{-1}{n+1}}-1}\frac{r^{\frac{i-1}{n+1}}-r}{r^{\frac{i}{n+1}}-r}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p$$
$$-\frac{1}{1-r^{\frac{1}{n+1}}}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p-\frac{r}{r^{\frac{i}{n+1}}-r}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\left(\frac{n+2-i}{n+1}\right)p_L\Delta p$$
$$+\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p$$
$$=\frac{1}{1-r^{\frac{1}{n+1}}}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p-\frac{1}{1-r^{\frac{1}{n+1}}}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p$$
$$+\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p-\frac{1}{r^{\frac{-1}{n+1}}-1}\frac{r^{\frac{i-1}{n+1}}-r}{r^{\frac{i}{n+1}}-r}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p$$
$$+\frac{r}{r^{\frac{i}{n+1}}-r}\left(\frac{n+1-i}{n+1}\right)p_L\Delta p-\frac{r}{r^{\frac{i}{n+1}}-r}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\left(\frac{n+2-i}{n+1}\right)p_L\Delta p$$

$$= \frac{1}{1-r^{\frac{1}{n+1}}}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}-\frac{1}{n+1}\right)\Delta p-\frac{1}{1-r^{\frac{1}{n+1}}}\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p$$

$$+\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}+\frac{1}{n+1}\right)\Delta p-\frac{1}{r^{\frac{-1}{n+1}}-1}\frac{r^{\frac{i-1}{n+1}}-r}{r^{\frac{i}{n+1}}-r}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p$$

$$+\overbrace{\frac{r}{r^{\frac{i}{n+1}}-r}\left(\frac{n+1-i}{n+1}\right)p_L\Delta p-\frac{r}{r^{\frac{i-1}{n+1}}-r}\left(\frac{n+2-i}{n+1}\right)p_L\Delta p}^{\gamma_i} \tag{B.4}$$

Now, we first simplify the first four terms in (B.4) as

$$\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{1}{n+1}}}\left(1-\frac{r^{\frac{i}{n+1}}-r}{r^{\frac{i-1}{n+1}}-r}\right)-\frac{\Delta p}{n+1}\frac{1}{1-r^{\frac{1}{n+1}}}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)$$

$$+\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p\frac{1}{r^{\frac{-1}{n+1}}-1}\left(1-\frac{r^{\frac{i-1}{n+1}}-r}{r^{\frac{i}{n+1}}-r}\right)+\frac{\Delta p}{n+1}\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)$$

$$=\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n+2-i}{n+1}}}-\frac{\Delta p}{n+1}\frac{1}{1-r^{\frac{1}{n+1}}}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)$$

$$-\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n+1-i}{n+1}}}+\frac{\Delta p}{n+1}\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)$$

$$=\overbrace{\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)\left(\frac{n+2-i}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n+2-i}{n+1}}}-\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)\left(\frac{n+1-i}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n+1-i}{n+1}}}}^{A_i}$$

$$\overbrace{-\frac{\Delta p}{n+1}\frac{1}{1-r^{\frac{1}{n+1}}}\left(\frac{n+2-i}{n+1}\Delta p+p_L\right)+\frac{\Delta p}{n+1}\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n+1-i}{n+1}\Delta p+p_L\right)}^{B_i}$$

Therefore, we have $\pi_i(p_i,\lambda_i')=A_i+B_i+\gamma_i$. To find $\pi(n)=\sum_{i=1}^{n}\pi_i$, we sum A_i, B_i, and γ_i from $i=1$ to n separately below.

$$\sum_{i=1}^{n}A_i=\left(\frac{n+1}{n+1}\Delta p+p_L\right)\left(\frac{n+1}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n+1}{n+1}}}-\left(\frac{n}{n+1}\Delta p+p_L\right)\left(\frac{n}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n}{n+1}}}$$

$$+\left(\frac{n}{n+1}\Delta p+p_L\right)\left(\frac{n}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n}{n+1}}}-\left(\frac{n-1}{n+1}\Delta p+p_L\right)\left(\frac{n-1}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n-1}{n+1}}}$$

$$+\left(\frac{n-1}{n+1}\Delta p+p_L\right)\left(\frac{n-1}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n-1}{n+1}}}-\left(\frac{n-2}{n+1}\Delta p+p_L\right)\left(\frac{n-2}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n-2}{n+1}}}$$

$$\cdots\cdots$$

$$+\left(\frac{3}{n+1}\Delta p+p_L\right)\left(\frac{3}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{3}{n+1}}}-\left(\frac{2}{n+1}\Delta p+p_L\right)\left(\frac{2}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{2}{n+1}}}$$

$$+\left(\frac{2}{n+1}\Delta p+p_L\right)\left(\frac{2}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{2}{n+1}}}-\left(\frac{1}{n+1}\Delta p+p_L\right)\left(\frac{1}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{1}{n+1}}}$$

$$=\left(\frac{n+1}{n+1}\Delta p+p_L\right)\left(\frac{n+1}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{n+1}{n+1}}}-\left(\frac{1}{n+1}\Delta p+p_L\right)\left(\frac{1}{n+1}\right)\Delta p\frac{1}{1-r^{\frac{1}{n+1}}}$$

$$=(\Delta p+p_L)\Delta p\frac{1}{1-r}-\left(\frac{\Delta p}{n+1}+p_L\right)\left(\frac{\Delta p}{n+1}\right)\frac{1}{1-r^{\frac{1}{n+1}}}$$

$$= \frac{p_H \Delta p}{1-r} - \frac{\left(\frac{\Delta p}{n+1} + p_L\right)\left(\frac{\Delta p}{n+1}\right)}{1-r^{\frac{1}{n+1}}}$$

$$\begin{aligned}
\sum_{i=1}^{n} B_i &= \sum_{i=1}^{n}\left(-\frac{\Delta p}{n+1}\frac{1}{1-r^{\frac{1}{n+1}}}\left(\frac{n+2-i}{n+1}\Delta p + p_L\right) + \frac{\Delta p}{n+1}\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n+1-i}{n+1}\Delta p + p_L\right)\right) \\
&= -\frac{\Delta p}{n+1}\frac{1}{1-r^{\frac{1}{n+1}}}\sum_{i=1}^{n}\left(\frac{n+2-i}{n+1}\Delta p + p_L\right) + \frac{\Delta p}{n+1}\frac{1}{r^{\frac{-1}{n+1}}-1}\sum_{i=1}^{n}\left(\frac{n+1-i}{n+1}\Delta p + p_L\right) \\
&= -\frac{\Delta p}{n+1}\frac{1}{1-r^{\frac{1}{n+1}}}\left(\left(\frac{n}{2} + \frac{n}{n+1}\right)\Delta p + np_L\right) + \frac{\Delta p}{n+1}\frac{1}{r^{\frac{-1}{n+1}}-1}\left(\frac{n}{2}\Delta p + np_L\right) \\
&= \frac{\Delta p}{n+1}\left(\frac{-1}{1-r^{\frac{1}{n+1}}}\left(\frac{n}{2}\Delta p + np_L\right) + \frac{n}{1+n}\Delta p\frac{-1}{1-r^{\frac{1}{n+1}}} + \frac{r^{\frac{1}{n+1}}}{1-r^{\frac{1}{n+1}}}\left(\frac{n}{2}\Delta p + np_L\right)\right) \\
&= \frac{\Delta p}{n+1}\left(-\left(\frac{n}{2}\Delta p + np_L\right) - \frac{n}{1+n}\frac{1}{1-r^{\frac{1}{n+1}}}\Delta p\right) \\
&= -\frac{n}{n+1}\left(\frac{\Delta p^2}{2} + p_L\Delta p + \Delta p^2\frac{\frac{1}{n+1}}{1-r^{\frac{1}{n+1}}}\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n}\gamma_i &= \sum_{i=1}^{n}\left(p_L\Delta p\frac{r}{r^{\frac{i}{n+1}}-r}\left(\frac{n+1-i}{n+1}\right) - p_L\Delta p\frac{r}{r^{\frac{i-1}{n+1}}-r}\left(\frac{n+2-i}{n+1}\right)\right) \\
&= p_L\Delta p\frac{r}{r^{\frac{1}{n+1}}-r}\left(\frac{n}{n+1}\right) - p_L\Delta p\frac{r}{r^{\frac{0}{n+1}}-r}\left(\frac{n+1}{n+1}\right) \\
&\quad + p_L\Delta p\frac{r}{r^{\frac{2}{n+1}}-r}\left(\frac{n-1}{n+1}\right) - p_L\Delta p\frac{r}{r^{\frac{1}{n+1}}-r}\left(\frac{n}{n+1}\right) \\
&\quad + p_L\Delta p\frac{r}{r^{\frac{3}{n+1}}-r}\left(\frac{n-2}{n+1}\right) - p_L\Delta p\frac{r}{r^{\frac{2}{n+1}}-r}\left(\frac{n-1}{n+1}\right) \\
&\quad \cdots\cdots \\
&\quad + p_L\Delta p\frac{r}{r^{\frac{n-1}{n+1}}-r}\left(\frac{2}{n+1}\right) - p_L\Delta p\frac{r}{r^{\frac{n-2}{n+1}}-r}\left(\frac{3}{n+1}\right) \\
&\quad + p_L\Delta p\frac{r}{r^{\frac{n}{n+1}}-r}\left(\frac{1}{n+1}\right) - p_L\Delta p\frac{r}{r^{\frac{n-1}{n+1}}-r}\left(\frac{2}{n+1}\right) \\
&= p_L\Delta p\frac{r}{r^{\frac{n}{n+1}}-r}\left(\frac{1}{n+1}\right) - p_L\Delta p\frac{r}{r^{\frac{0}{n+1}}-r}\left(\frac{n+1}{n+1}\right) \\
&= p_L\Delta p\left(\frac{1}{n+1}\frac{1}{r^{\frac{-1}{n+1}}-1} - \frac{r}{1-r}\right)
\end{aligned}$$

Using $\pi(n) = \sum_{i=1}^{n}\pi_i = \sum_{i=1}^{n}(A_i + B_i + \gamma_i)$, we have

$$\pi(n) = \frac{p_H\Delta p}{1-r} - \frac{\frac{\Delta p}{n+1}\left(\frac{\Delta p}{n+1} + p_L\right)}{1-r^{\frac{1}{n+1}}} - \frac{n}{n+1}\left(\frac{\Delta p^2}{2} + p_L\Delta p + \Delta p^2\frac{\frac{1}{n+1}}{1-r^{\frac{1}{n+1}}}\right) + p_L\Delta p\left(\frac{\frac{1}{n+1}}{r^{-\frac{1}{n+1}}-1} - \frac{r}{1-r}\right)$$

■.

Proof of Proposition 7.

When the seller offers n PS goods, the optimal pricing problem is as follows.

$$\max_{p_H,p_L,p_i,\lambda_i,i\in\{1,\ldots,n\}} \Pi(p_H,p_L,p_1,\ldots,p_n,\lambda_1,\ldots,\lambda_n) =$$
$$(p_H-c_H)(1-w_{0,1})+(p_L-c_L)(w_{n,n+1}-\tau p_L)+\sum_{i=1}^{n}\big(p_i-\lambda_i c_H-(1-\lambda_i)c_L\big)(w_{i-1,i}-w_{i,i+1})$$

Similar to how we solved the optimal component-good pricing problem when a single probabilistic good is offered, we solve the above problem using backward induction. Thanks to Lemma 2, we have the optimal p_i and λ_i given below:

$$\lambda_i=\frac{r^{\frac{i}{n+1}}-r}{1-r},\qquad p_i=\frac{n+1-i}{n+1}p_H+\frac{i}{n+1}p_L.$$

with $p_0=p_H$ and $p_{n+1}=p_L$, $\lambda_0=1$ and $\lambda_{n+1}=0$.

From Equation (10), we have

$$\begin{aligned}
w_{i-1,i}=&\frac{(\lambda_{i-1}(q_H-q_L)+q_L)p_{i-1}-(\lambda_i(q_H-q_L)+q_L)p_i}{(\lambda_{i-1}-\lambda_i)(q_H-q_L)}\\
=&\frac{(\lambda_{i-1}+\gamma-1)p_{i-1}-(\lambda_i+\gamma-1)p_i}{\lambda_{i-1}-\lambda_i}\\
=&\frac{\lambda_{i-1}+\gamma-1}{\lambda_{i-1}-\lambda_i}\left(\frac{n+2-i}{n+1}p_H+\frac{i-1}{n+1}p_L\right)-\frac{\lambda_i+\gamma-1}{\lambda_{i-1}-\lambda_i}\left(\frac{n+1-i}{n+1}p_H+\frac{i}{n+1}p_L\right)
\end{aligned}$$

In particular, we have

$$\begin{aligned}
w_{0,1}&=\frac{\gamma}{1-\lambda_1}p_H-\frac{\lambda_1+\gamma-1}{1-\lambda_1}\left(\frac{n}{n+1}p_H+\frac{1}{n+1}p_L\right)\\
w_{n,n+1}&=\frac{\lambda_n+\gamma-1}{\lambda_n}\left(\frac{1}{n+1}p_H+\frac{n}{n+1}p_L\right)-\frac{\gamma-1}{\lambda_n}p_L.
\end{aligned}$$

To express the profit as a function of p_H and p_L , we calculate $w_{i-1,i}-w_{i,i+1}$ below.

$$\begin{aligned}
w_{i-1,i}-w_{i,i+1}=&\frac{\lambda_{i-1}+\gamma-1}{\lambda_{i-1}-\lambda_i}\left(\frac{n+2-i}{n+1}p_H+\frac{i-1}{n+1}p_L\right)-\frac{\lambda_i+\gamma-1}{\lambda_{i-1}-\lambda_i}\left(\frac{n+1-i}{n+1}p_H+\frac{i}{n+1}p_L\right)\\
&-\frac{\lambda_i+\gamma-1}{\lambda_i-\lambda_{i+1}}\left(\frac{n+1-i}{n+1}p_H+\frac{i}{n+1}p_L\right)+\frac{\lambda_{i+1}+\gamma-1}{\lambda_i-\lambda_{i+1}}\left(\frac{n-i}{n+1}p_H+\frac{i+1}{n+1}p_L\right)\\
=&\left(\frac{\lambda_{i-1}+\gamma-1}{\lambda_{i-1}-\lambda_i}\frac{n+2-i}{n+1}-\frac{\lambda_i+\gamma-1}{\lambda_{i-1}-\lambda_i}\frac{n+1-i}{n+1}-\frac{\lambda_i+\gamma-1}{\lambda_i-\lambda_{i+1}}\frac{n+1-i}{n+1}+\frac{\lambda_{i+1}+\gamma-1}{\lambda_i-\lambda_{i+1}}\frac{n-i}{n+1}\right)p_H\\
&+\left(\frac{\lambda_{i-1}+\gamma-1}{\lambda_{i-1}-\lambda_i}\frac{i-1}{n+1}-\frac{\lambda_i+\gamma-1}{\lambda_{i-1}-\lambda_i}\frac{i}{n+1}-\frac{\lambda_i+\gamma-1}{\lambda_i-\lambda_{i+1}}\frac{i}{n+1}+\frac{\lambda_{i+1}+\gamma-1}{\lambda_i-\lambda_{i+1}}\frac{i+1}{n+1}\right)p_L\\
=&\frac{1}{n+1}(p_H-p_L).
\end{aligned}$$

Note that in the last step, we used the fact that for any i,

$$\frac{\lambda_i+\gamma-1}{\lambda_i-\lambda_{i+1}}=\frac{\frac{r^{\frac{i}{n+1}}-r}{1-r}+\frac{1}{1-r}-1}{\frac{r^{\frac{i}{n+1}}-r}{1-r}-\frac{r^{\frac{i+1}{n+1}}-r}{1-r}}=\frac{1}{1-r^{\frac{1}{n+1}}}$$

$$\frac{\lambda_{i+1}+\gamma-1}{\lambda_i-\lambda_{i+1}}=\frac{\lambda_i+\gamma-1}{\lambda_i-\lambda_{i+1}}-1=\frac{r^{\frac{1}{n+1}}}{1-r^{\frac{1}{n+1}}}.$$

Therefore, we can rewrite the objective function as

$$\begin{aligned}\Pi(p_H,p_L,p_1,...,p_n,\lambda_1,...,\lambda_n)= \quad &(p_H-c_H)\left(1-\tfrac{\gamma}{1-\lambda_1}p_H+\tfrac{\lambda_1+\gamma-1}{1-\lambda_1}\left(\tfrac{n}{n+1}p_H+\tfrac{1}{n+1}p_L\right)\right)\\ &+(p_L-c_L)\left(\tfrac{\lambda_n+\gamma-1}{\lambda_n}\left(\tfrac{1}{n+1}p_H+\tfrac{n}{n+1}p_L\right)-\tfrac{\gamma-1}{\lambda_n}p_L-\tau p_L\right)\\ &+\tfrac{p_H-p_L}{n+1}\textstyle\sum_{i=1}^n\left(\tfrac{n+1-i}{n+1}p_H+\tfrac{i}{n+1}p_L-\lambda_i c_H-(1-\lambda_i)c_L\right)\end{aligned}$$

where we used the fact $\underline{w}=\tau p_L$ with $\tau\equiv\frac{q_L}{q_L-q_0}$.

Taking partial derivative with respect to p_H, we have

$$\begin{aligned}\frac{\partial\Pi}{\partial p_H}=&\left(1-\frac{\gamma}{1-\lambda_1}p_H+\frac{\lambda_1+\gamma-1}{1-\lambda_1}\left(\frac{n}{n+1}p_H+\frac{1}{n+1}p_L\right)\right)+(p_H-c_H)\left(-\frac{\gamma}{1-\lambda_1}+\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{n}{n+1}\right)\\ &+(p_L-c_L)\frac{\lambda_n+\gamma-1}{\lambda_n}\frac{1}{n+1}\\ &+\sum_{i=1}^n\left(\frac{n+1-i}{n+1}(\frac{1}{n+1}p_H-\frac{1}{n+1}p_L)+\left(\frac{n+1-i}{n+1}p_H+\frac{i}{n+1}p_L-\lambda_i c_H-(1-\lambda_i)c_L\right)\frac{1}{n+1}\right)\\ =&\left(\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{n}{n+1}-\frac{\gamma}{1-\lambda_1}-\frac{\gamma}{1-\lambda_1}+\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{n}{n+1}+\sum_{i=1}^n 2\frac{n+1-i}{n+1}\frac{1}{n+1}\right)p_H\\ &+\left(\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{1}{n+1}+\frac{\lambda_n+\gamma-1}{\lambda_n}\frac{1}{n+1}+\sum_{i=1}^n\left(-\frac{n+1-i}{n+1}\frac{1}{n+1}+\frac{i}{n+1}\frac{1}{n+1}\right)\right)p_L\\ &+1-c_H\left(-\frac{\gamma}{1-\lambda_1}+\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{n}{n+1}\right)-c_L\frac{\lambda_n+\gamma-1}{\lambda_n}\frac{1}{n+1}-\sum_{i=1}^n\left(\lambda_i c_H+(1-\lambda_i)c_L\right)\frac{1}{n+1}\end{aligned}$$

Setting $\partial\Pi/\partial p_H=0$, we have

$$\begin{aligned}&\underbrace{2\left(\frac{n\lambda_1-n-\gamma}{1-\lambda_1}+\sum_{i=1}^n\frac{n+1-i}{n+1}\right)}_{A_{11}}p_H^*+\underbrace{\left(\frac{\gamma(\lambda_n-\lambda_1)+\gamma-1+\lambda_1}{(1-\lambda_1)\lambda_n}-\sum_{i=1}^n\left(\frac{n+1-2i}{n+1}\right)\right)}_{A_{12}}p_L^*\\ &=\underbrace{\left(\sum_{i=1}^n\lambda_i+\frac{n\lambda_1-n-\gamma}{1-\lambda_1}\right)c_H+\left(\sum_{i=1}^n(1-\lambda_i)+\frac{\lambda_n+\gamma-1}{\lambda_n}\right)c_L-(n+1)}_{A_{13}}\end{aligned}$$

Note that

$$\lambda_i=\frac{r^{\frac{i}{n+1}}-r}{1-r},\qquad \gamma=\frac{1}{1-r}.$$

We can simplify A_{11} and A_{12}

$$\begin{aligned}A_{11}&=2\frac{n\lambda_1-n-\gamma}{1-\lambda_1}+n=-n-\frac{2\gamma}{1-\lambda_1}=-n-\frac{2}{1-r^{\frac{1}{n+1}}}\\ A_{12}&=\frac{\gamma(\lambda_n-\lambda_1)+\gamma-1+\lambda_1}{(1-\lambda_1)\lambda_n}=1+\frac{2r^{\frac{1}{n+1}}}{1-r^{\frac{1}{n+1}}}\end{aligned}$$

It's easy to see that $A_{11}+A_{12}=-n-1$. Noting that

$$\sum_{i=1}^{n}\lambda_i=\frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}-\frac{nr}{1-r},\qquad \sum_{i=1}^{n}(1-\lambda_i)=\frac{n}{1-r}-\frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}},$$

we can write A_{13} as

$$A_{13}=\left(\frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}-\frac{n}{1-r}-\frac{1}{1-r^{\frac{1}{n+1}}}\right)c_H+\left(\frac{n}{1-r}-\frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}+\frac{1}{1-r^{\frac{1}{n+1}}}\right)c_L-(n+1)$$

Taking partial derivative with respect to p_L, we have

$$\begin{aligned}\frac{\partial\Pi}{\partial p_L}=&(p_H-c_H)\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{1}{n+1}\\&+\frac{\lambda_n+\gamma-1}{\lambda_n}\left(\frac{1}{n+1}p_H+\frac{n}{n+1}p_L\right)-\frac{\gamma-1}{\lambda_n}p_L-\tau p_L+(p_L-c_L)\left(\frac{\lambda_n+\gamma-1}{\lambda_n}\frac{n}{n+1}-\frac{\gamma-1}{\lambda_n}-\tau\right)\\&+\sum_{i=1}^{n}\left(\frac{i}{n+1}\left(\frac{1}{n+1}p_H-\frac{1}{n+1}p_L\right)-\left(\frac{n+1-i}{n+1}p_H+\frac{i}{n+1}p_L-\lambda_i c_H-(1-\lambda_i)c_L\right)\frac{1}{n+1}\right)\\=&\left(\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{1}{n+1}+\frac{\lambda_n+\gamma-1}{\lambda_n}\frac{1}{n+1}+\sum_{i=1}^{n}\left(\frac{i}{n+1}\frac{1}{n+1}-\frac{n+1-i}{n+1}\frac{1}{n+1}\right)\right)p_H\\&+2\left(\frac{\lambda_n+\gamma-1}{\lambda_n}\frac{n}{n+1}-\frac{\gamma-1}{\lambda_n}-\tau-\sum_{i=1}^{n}\frac{i}{n+1}\frac{1}{n+1}\right)p_L-\left(\frac{\lambda_1+\gamma-1}{1-\lambda_1}\frac{1}{n+1}-\sum_{i=1}^{n}\lambda_i\frac{1}{n+1}\right)c_H\\&-\left(\frac{\lambda_n+\gamma-1}{\lambda_n}\frac{n}{n+1}-\frac{\gamma-1}{\lambda_n}-\tau-\sum_{i=1}^{n}\frac{1-\lambda_i}{n+1}\right)c_L\end{aligned}$$

Setting $\partial\Pi/\partial p_L=0$, we have

$$\begin{aligned}&\underbrace{\left(\frac{\gamma(\lambda_n-\lambda_1)+\gamma-1+\lambda_1}{(1-\lambda_1)\lambda_n}-\sum_{i=1}^{n}\left(\frac{n+1-2i}{n+1}\right)\right)}_{A_{21}}p_H^*+2\underbrace{\left(\frac{n\lambda_n-\gamma+1}{\lambda_n}-\tau(n+1)-\sum_{i=1}^{n}\frac{i}{n+1}\right)}_{A_{22}}p_L^*\\=&\underbrace{\left(\frac{\lambda_1+\gamma-1}{1-\lambda_1}-\sum_{i=1}^{n}\lambda_i\right)c_H+\left(\frac{n\lambda_n-\gamma+1}{\lambda_n}-(n+1)\tau-\sum_{i=1}^{n}(1-\lambda_i)\right)c_L}_{A_{23}}\end{aligned}$$

We can simplify A_{21}, A_{22} and A_{23} as

$$\begin{aligned}A_{21}&=\frac{\gamma(\lambda_n-\lambda_1)+\gamma-1+\lambda_1}{(1-\lambda_1)\lambda_n}=A_{12}\\A_{22}&=n-\frac{2\gamma-2}{\lambda_n}-2\tau(n+1)\neq A_{11}\\A_{23}&=\left(\frac{\lambda_1+\gamma-1}{1-\lambda_1}-\sum_{i=1}^{n}\lambda_i\right)c_H+\left(\frac{1-\lambda_n-\gamma}{\lambda_n}+(n+1)(1-\tau)-\sum_{i=1}^{n}(1-\lambda_i)\right)c_L\\&=\left(\frac{r^{\frac{1}{n+1}}}{1-r^{\frac{1}{n+1}}}+\frac{nr}{1-r}-\frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}\right)c_H-\left(\frac{n}{1-r}-\frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}+\frac{1}{1-r^{\frac{1}{n+1}}}-(n+1)(1-\tau)\right)c_L\end{aligned}$$

$$\zeta \equiv \frac{2}{1-r^{\frac{1}{n+1}}} \Longrightarrow A_{11} = -(\zeta+n), \quad A_{22} = -(\zeta+n) - 2(\tau-1)(n+1) \quad A_{12} = A_{21} = \zeta - 1.$$

It remains to solve the following system of linear equations:

$$\begin{aligned}
(\zeta+n)p_H^* - (\zeta-1)p_L^* &= n+1-\Big(\frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}} - \frac{n}{1-r} - \frac{\zeta}{2}\Big)c_H \\
&\quad -\Big(\frac{n}{1-r} - \frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}} + \frac{\zeta}{2}\Big)c_L \\
(\zeta-1)p_H^* - \big((\zeta+n)+2(\tau-1)(n+1)\big)p_L^* &= \Big(\frac{\zeta}{2} - 1 + \frac{nr}{1-r} - \frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}\Big)c_H \\
&\quad -\Big(\frac{n}{1-r} - \frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}} + \frac{\zeta}{2} - (n+1)(1-\tau)\Big)c_L
\end{aligned}$$

Subtracting the second equation from the first one and dividing both sides by $(n+1)$, we obtain

$$p_H^* + (2\tau-1)p_L^* = 1 + c_H - (1-\tau)c_L.$$

Substituting $p_L^* = \frac{1+c_H-(1-\tau)c_L-p_H^*}{2\tau-1}$ into the first equation, we obtain

$$\begin{aligned}
p_H^* &= \frac{1}{\zeta(1+\frac{1}{2\tau-1}) + n - \frac{1}{2\tau-1}}\left(\frac{\zeta-1}{2\tau-1} + n + 1 + c\Big(\frac{n}{1-r} + \frac{\zeta}{2} - \frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}\Big) + c_H\frac{\zeta-1}{2\tau-1}\right) \\
p_L^* &= \frac{1+c_H-(1-\tau)c_L-p_H^*}{2\tau-1} = \frac{1+c_H-(1-\tau)c_L}{2\tau-1} \\
&\quad - \frac{1}{(2\tau-1)\big(\zeta(1+\frac{1}{2\tau-1}) + n - \frac{1}{2\tau-1}\big)}\left(\frac{\zeta-1}{2\tau-1} + n + 1 + c\Big(\frac{n}{1-r} + \frac{\zeta}{2} - \frac{1-r^n}{(1-r)^2}r^{\frac{1}{n+1}}\Big) + c_H\frac{\zeta-1}{2\tau-1}\right)\Bigg)
\end{aligned}$$

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