Probabilistic Selling in Vertically Differentiated Markets

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Outline

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Hotel

The **Tenerife Roulette** is an open hotel booking system that guarantees accommodation in one of the following 4* and 5* hotels on the island of Tenerife: H10 Conquistador, H10 Costa Adeje Palace, H10 Las Palmeras, H10 Big Sur (Adults Only), H10 Gran Tinerfe (Adults Only) and H10 Atlantic Sunset. All the hotels face the sea and have spacious swimming pools and renovated spaces with a contemporary interior design. Our hotels in Tenerife offer a wide range of dining options and a complete entertainment programme for the whole family.

Formula Roulette Prestige

A Delphina offer for your Hotel holiday in Sardinia

Discounts up to 60% at the 4 and 5 star Delphina hotels by the fabulous sea in Sardinia

The Delphina Prestige Roulette Formula is a guarantee of quality and savings. Book your holiday with **Delphina** now and you can stay in a **4 star hotel or in a 5 star hotel** in **Sardinia.** You can always be sure of counting on **Delphina**'s constant high quality together with discounts of up to 60%.

Literature

- **Horizontal**: morning vs. afternoon flight, green vs. red sweater
- **Vertical**: business vs. economy, 5-star vs. 4-star, high- vs low-speed

	Rational	Bounded Rationality
Horizontal	Jiang (2007), Fay and Xie (2008,2010), Jerath et al. (2010,2009), Shapiro and Shi (2008)	Huang and Yu (2014)
Vertical	Biyalagorsky (2005), Zhang et al. (2015), current paper	Huang and Yu (2014), Zheng et al. (2019)

Probabilistic selling is theoretically

- well-justified for horizontally differentiated markets.
- never profitable in verticall differentiated markets unless one introduces asymmetric capacity constraints or bounded rationality.

$\theta q - p$: What is the Nature of θ?

This utility function, equivalent to $q - \theta p$ if we re-define θ by its reciprocal, can be motivated (Tirole, 1988) by approximating

$$U(w-p, q) \equiv u(w-p)+q = u(w) - pu'(\hat{w}) + q \quad \text{where } w-p < \hat{w} < w.$$

Because $u(w)$ plays no role in the consumer's choice problem, the above utility is equivalent to

$$q - \underbrace{u'(\hat{w})}_{\theta} p$$

Can we interpret θ as consumer type?

- yes, but only in the limit as $p \to 0$.
- because θ is a function of **both** w and p.

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$U(x, y)$: a consumer's preference over the consumption of

- money (x) which is interpreted as Hicks' composite good, and
- the focal good (y).

Focal Good: Indivisible with Unit Demand

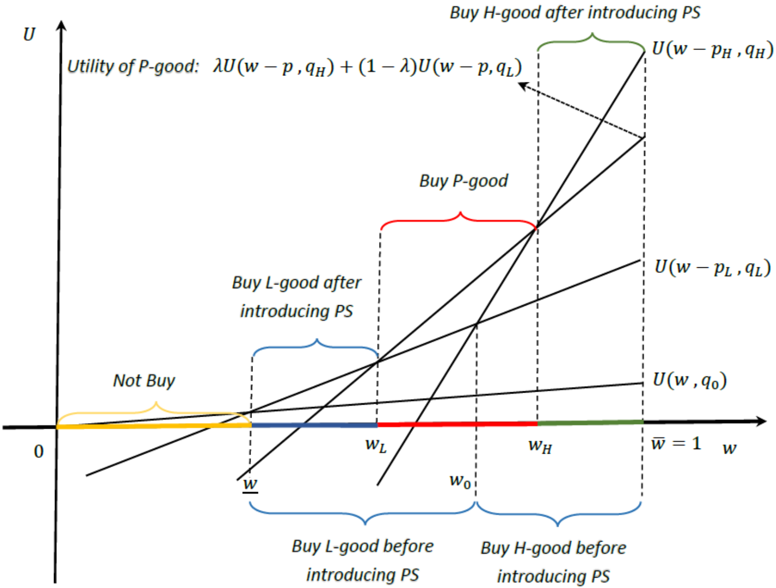
- Component goods:

$$(\text{price, quality, cost}) = \begin{cases} (p_H, q_H, c_H) \\ (p_L, q_L, c_L) \end{cases}$$

- Probabilistic good (p, λ):

consumer expected utility $= \lambda U(w-p, q_H) + (1-\lambda) U(w-p, q_L)$

Regularity Conditions: $\dfrac{\partial^2 U}{\partial x \partial y} > 0, \quad \dfrac{\partial^2 U}{\partial x^2} \leq 0, \quad \dfrac{\partial^2 U}{\partial y^2} \leq 0$


Buy H-good after introducing PS
U
Utility of P-good: $\lambda U(w-p,q_H)+(1-\lambda)U(w-p,q_L)$
$U(w-p_H,q_H)$
Buy P-good
Buy L-good after
introducing PS
$U(w-p_L,q_L)$
Not Buy
$U(w,q_0)$
0
w_L
w_H
$\bar{w}=1$
w
$\underline{w}$
w_0
Buy L-good before
introducing PS
Buy H-good before
introducing PS

Pivotal consumers: those with budget level w_0 where

$$U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L)$$

Pivotal Consumers

The demand for the probabilistic good (p, λ) is positive if and only if the pivotal consumer strictly prefers the probabilistic good (p, λ) to either component good.

Equivalently, the demand for probabilistic good (p, λ) is positive if and only if $p < \bar{p}(\lambda)$ where $\bar{p}(\lambda)$ uniquely solves the following equation of p:

$$\begin{aligned} & \lambda U(w_0 - p, q_H) + (1-\lambda) U(w_0 - p, q_L) \\ = \quad & U(w_0 - p_H, q_H) = U(w_0 - p_L, q_L) \end{aligned}$$

Moreover, $\bar{p}(\lambda) > \lambda p_H + (1-\lambda) p_L$.

Let (p_h^*, p_l^*) be the optimal prices of the component good without probabilistic selling.

The Sufficiency of λ-Concavity

Probabilistic selling is profitable if consumer preference and budget distribution are λ-concave for some $\lambda \in (0,1)$ at (p_h^*, p_l^*).

Let $\succeq$ be a two-attribute preference represented by the utility function $U(x, y)$ that satisfies the regularity conditions, and F be a cumulative distribution function. The pair $(\succeq, F)$ is called λ-concave for $\lambda \in (0,1)$, if

$$F(y) > \lambda F(x) + (1-\lambda)F(z) \iff \frac{F(z) - F(y)}{F(y) - F(x)} < \frac{\lambda}{1-\lambda}.$$

where $x < y < z$ are unique solutions to the following equations:

$$U(x - p_L, q_L) = \lambda U\big(x - \lambda p_H - (1-\lambda)p_L, q_H\big) + (1-\lambda)U\big(x - \lambda p_H - (1-\lambda)p_L, q_L\big)$$

$$U(y - p_H, q_H) = U(y - p_L, q_L)$$

$$U(z - p_H, q_H) = \lambda U\big(z - \lambda p_H - (1-\lambda)p_L, q_H\big) + (1-\lambda)U\big(z - \lambda p_H - (1-\lambda)p_L, q_L\big).$$

Example 1: Linear Preference

Consider the linear utility function $\theta q - p$ and interpret F as the distribution of θ. It's straightforward to verify that

$$y = \frac{p_H - p_L}{q_H - q_L}, \quad x = \frac{\lambda p_H + (1-\lambda)p_L - p_L}{\lambda(q_H - q_L)} = y, \quad z = \frac{p_H - \lambda p_H - (1-\lambda)p_L}{(1-\lambda)(q_H - q_L)} = y$$

Because $x = y = z$, the linear utility function is not λ-concave with any distribution.

Example 2: Cobb-Douglas Utility

The family of Cobb-Douglas utility functions and the uniform distribution are 1/2-concave for any (p_H, p_L, q_H, q_L). Hence, if p_h^* and p_l^* are the optimal prices of the component goods without probabilistic selling, the probabilistic selling strategy $(p_h^*, p_l^*, \frac{p_h^*+p_l^*}{2}, \frac{1}{2})$ can increase the profit.

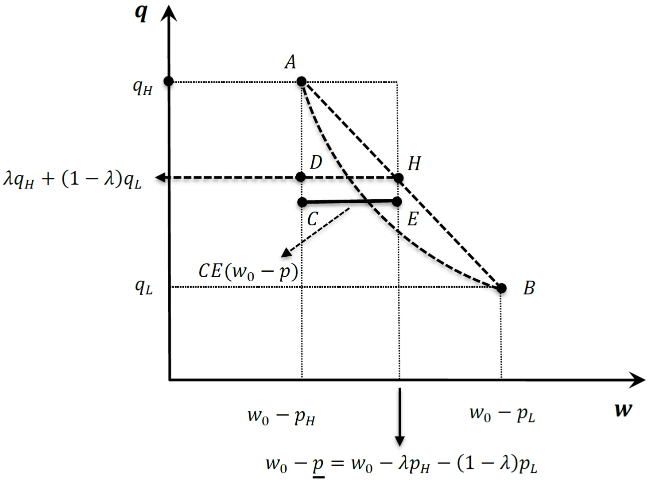
The Almost Sufficiency of Preference Convexity

Given any $\lambda \in (0,1)$ and any strictly quasiconcave utility function satisfying the regularity conditions, there exists a distribution such that it is λ-concave with the preference. Such a distribution can always be chosen as absolutely continuous.

$$\frac{F(z) - F(y)}{F(y) - F(x)} < \frac{\lambda}{1-\lambda}$$

- For any strictly quasiconcave utility function satisfying the regularity conditions and any distribution, does there exist a $\lambda \in (0,1)$ such that the pair is λ-concave?
- If not, what are the requirements on the utility function and/or the distribution?

The Importance of Preference Convexity



The preference relation $\succeq$ on $\mathcal{X}$ is convex if for every $x \in \mathcal{X}$, the upper contour set $\{y \in \mathcal{X} : y \succeq x\}$ is convex.

A rational preference is strictly convex if and only if it can be represented by a strictly quasiconcave utility function.

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3 Optimal Design & Market Implications

How to Design Probabilistic Selling?

Let w_H and w_L be the unique solutions to the following equations:

$$U(w_H - p_H, q_H) = \lambda U(w_H - p, q_H) + (1-\lambda) U(w_H - p, q_L) \quad (1)$$
$$U(w_L - p_L, q_L) = \lambda U(w_L - p, q_H) + (1-\lambda) U(w_L - p, q_L) \quad (2)$$

$$p < \bar{p}(\lambda) \Longrightarrow w_L < w_0 < w_H.$$

- $\pi^0(p_H, p_L)$: seller's profit of offering only 2 component goods
- $\Pi(p_H, p_L, p, \lambda)$: seller's profit of offering both the component goods and the probabilistic goods

Assume uniform distribution of w with support $[0, 1]$.

Problem 1 (No PS): Let (p_h^*, p_l^*) be the solution.

$$\max_{0<p_L<p_H<1} \pi^0(p_H, p_L)$$

Problem 2 (Design): Let $\big(p^*(p_H, p_L), \lambda^*(p_H, p_L)\big)$ be the solution.

$$\begin{aligned}\max_{p\leq\bar{p},\ \lambda\in(0,1)} \pi(p,\lambda) = \quad & \Big(p - \big(\lambda c_H + (1-\lambda)c_L\big)\Big)(w_H - w_L) \\ & -(p_H - c_H)(w_H - w_0) - (p_L - c_L)(w_0 - w_L).\end{aligned}$$

Problem 3 (PS): Let $\big(p_H^*, p_L^*, p^*(p_H^*, p_L^*), \lambda^*(p_H^*, p_L^*)\big)$ be the solution.

$$\max_{0<p_L<p_H<1} \Pi(p_H, p_L) \equiv \pi^0(p_H, p_L) + \pi^*(p_H, p_L).$$

Comparative Statics

$$\frac{\partial \pi^*(p_H, p_L)}{\partial p_H} + \frac{\partial \pi^*(p_H, p_L)}{\partial p_L} = 0.$$

Can Probabilistic Selling Increase Market Coverage?

If $p_L^* < p_l^*$, some consumers who previously could not afford the focal product can now afford. Those who would purchase the low-quality component good in the absence of probabilistic selling are also better off with the introduction of probabilistic selling.

Efficiency

Suppose $\lim_{x\to 0} U(x,y) = \lim_{y\to 0} U(x,y) = \underline{U}$ where $\underline{U}$ is the lower bound of the utility. If the following two conditions are satisfied,

$$\frac{\partial w_H}{\partial p_H}\left(2-\frac{\partial p^*}{\partial p_H}+c\frac{\partial \lambda^*}{\partial p_H}\right) > \frac{1}{2}+\left(p^*-p_H+(1-\lambda^*)c\right)\frac{\partial^*}{\partial p_H}\left(\frac{\partial w_H}{\partial p_H}\right), \forall (p_H, p_L),$$

$$(p^* - p_L - c\lambda^*)\frac{\partial w_L}{\partial p_L} > w_L + c_L - 2p_L \quad \text{at } (p_H, p_L) = (p_h^*, p_l^*),$$

the optimal price of the high-quality (low-quality) component good increases (decreases) upon the introduction of probabilistic selling, and by the same amount, i.e., $p_H^* - p_h^* = p_l^* - p_L^*$.

Example: $U(x,y) = xy$

Let $\gamma \equiv q_H/(q_H - q_L) > 1$.

$$w_0 = \gamma p_H + (1-\gamma) p_L, \quad \bar{p}(\lambda) = \frac{\lambda\gamma}{\gamma - 1 + \lambda} p_H + \left(1 - \frac{\lambda\gamma}{\gamma - 1 + \lambda}\right) p_L$$

$$w_H = \frac{p_H}{1-\lambda}\gamma - \frac{p}{1-\lambda}(\gamma - 1 + \lambda), \quad w_L = \frac{p}{\lambda}(\gamma - 1 + \lambda) + \frac{p_L}{\lambda}(1-\gamma).$$

Given (p_H, p_L), the optimal design of the probabilistic good is

$$p^* = \frac{p_H + p_L}{2}, \qquad \lambda^* = \sqrt{\gamma(\gamma-1)} - \gamma + 1 = \frac{\sqrt{q_L}}{\sqrt{q_H} + \sqrt{q_L}}$$

$$\pi^*(p_H, p_L) = \left(\frac{1}{2} - \lambda^*\right) \frac{2\lambda^*\gamma - \gamma + 1 - \lambda^*}{2\lambda^*(1-\lambda^*)} (p_H - p_L)^2$$

$$\frac{\gamma}{1-\lambda^*}\left(2 - \frac{1}{2} + c \cdot 0\right) > \frac{1}{2} + (p^* - p_H + (1-\lambda^*)c) \frac{\partial^*}{\partial p_H}\left(\frac{\gamma}{1-\lambda^*}\right) = \frac{1}{2} + 0$$

The optimal (p_H, p_L, p, λ) is

$$p_H^* = \frac{\kappa c_H + (1+\kappa) + \gamma(c_H - c_L)}{1+2\kappa}, \qquad p_L^* = \frac{(1+\kappa)c_H + \kappa - \gamma(c_H - c_L)}{1+2\kappa}$$

$$p^* = \frac{1+c_H}{2}, \qquad \lambda^* = \sqrt{\gamma(\gamma-1)} - \gamma + 1.$$

where $\kappa \equiv \frac{1}{2}(\sqrt{\gamma} - \sqrt{\gamma - 1})^{-2}$. The relative increases of market coverage is

$$\left(1 - \sqrt{1 - \frac{1}{\gamma}}\right)^3 \frac{1 - c_H + 2\gamma(c_H - c_L)}{4(2 - c_H - c_L)}.$$

By viewing the probabilistic good as another "component" good, we may design two more probabilistic goods, with one mixing the high-quality component good and the probabilistic good, and with the other mixing the low-quality component good and the probabilistic good.

Given (p_H, p_L), the optimal menu of n probabilistic goods consists of a sequence of probabilistic goods $\{\lambda_i, p_i\}$ where, $\forall i \in \{1, \cdots, n\}$,

$$p_i^* = \frac{n+1-i}{n+1} p_H + \frac{i}{n+1} p_L, \quad \lambda_i^* = \frac{r^{\frac{i}{n+1}} - r}{1-r}.$$

The optimal $(p_H, p_L, \lambda_1, \cdots, \lambda_n, p_1, \cdots, p_n)$ is

$$\begin{aligned}
p_H^* &= \frac{n + \zeta + (c_H - c_L)\left(\frac{n}{1-r} + \frac{\zeta}{2} - \frac{1-r^n}{(1-r)^2} r^{\frac{1}{n+1}}\right) + c_H(\zeta - 1)}{2\zeta + n - 1} \\
p_L^* &= 1 + c_H - p_H^* \\
p_i^* &= \frac{n+1-i}{n+1} p_H^* + \frac{i}{n+1} p_L^* \\
\lambda_i^* &= \frac{r^{\frac{i}{n+1}} - r}{1-r}
\end{aligned}$$

where $\zeta \equiv \frac{2}{1-r^{\frac{1}{n+1}}}$.

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Contributions to Literature

1. Identified the importance of preference convexity and the sufficiency of λ-concavity for the profitablity of probabilistic selling in vertically differentiated markets.
2. Developed a theory for optimal probabilistic selling.
3. Initiated the study of designing multiple probabilistic goods, a direction emphasized in the literature.

The drastically different finding obtained from strictly convex consumer preference suggests that linear approximation is not always without consequence. Analytical research can benefit from a robustness check with some alternative utility function.

- Surprising results are rarely robust;
- Robust results are often not surprising.

Managerial Implications

1. Because preference convexity is a widely accepted assumption in neoclassical economics, the potential of probabilistic selling is beyond what have been discussed in the current literature.
2. Under certain regularity conditions, the market coverage increases as a result of probabilistic selling. Therefore, probabilistic selling can not only improve profit, but can also increase efficiency.
3. The profit gain from probabilistic selling increases as the quality (or price) difference between two component goods increases, probabilistic selling is particularly appealing in market settings where quality (or price) difference between different goods is large.

Thank You!