Allocation and Pricing of Substitute Goods: Theory and Algorithm

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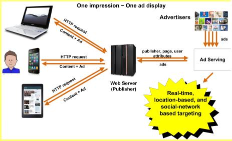
Outline

Substitute Goods/Tasks

- Substitute goods/tasks are goods/tasks which may replace each other in consumption or fulfillment.
- The advance in Internet technology and the growth of sharing economy has created massive number of substitute goods/tasks of different types.
- The demand for such substitute goods/tasks may also be massively heterogeneous.



Massively Substitute Goods

Objectives

- Provide a theory for pricing massive number of types of substitute goods given the heterogeneous preferences.
- Develop a scalable algorithm for efficiently allocating such goods and deriving the corresponding prices.

Negishi (1960)

A competitive equilibrium[a] is a maximum point of a social welfare function that consists of a linear combination of utility functions of consumers.

[a] A competitive equilibrium refers to an allocation and a set of prices such that the allocation is feasible and non-negative and all price-taking agents find their respective allocation optimal.

Outline

Goods, Agents, and Allocation

Goods

There are a set of M types of goods $\mathcal{M} = \{1, 2, \cdots, M\}$. The supply is $w = (w_1, w_2, \cdots, w_M)$.

Agents

There are a set of N agents $\mathcal{N} = \{1, 2, \cdots, N\}$ who value the M types of goods.

Allocation

$$\mathbf{x} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \vdots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}$$

Preferences and Constraints

Preferences

$$\mathcal{U}_i\left(x_i\right) = Q_i\left(\sum_{m=1}^{M} \alpha_{im} x_{im}\right), \ \forall i \in \mathcal{N} \tag{1}$$

Assumption: $Q_i(\cdot)$ is increasing and concave.

M Feasibility Constraint

$$\sum_{i \in \mathcal{N}} x_{im} \leq w_m, \ \forall m \in \mathcal{M}$$

$M \times N$ Non-negative Constraint

$$x_{im} \geq 0, \forall i \in \mathcal{N}, \ m \in \mathcal{M}$$

Nonlinear Allocation with Substitution (NAS)

$$\max_{\{x_{im}\}} \quad S = \sum_{i \in \mathcal{N}} \mathcal{U}_i(x_i) = \sum_{i \in \mathcal{N}} Q_i \left(\sum_{m \in \mathcal{M}} \alpha_{im} x_{im} \right) \tag{2}$$

$$s.t. \quad \sum_{i \in \mathcal{N}} x_{im} \le \omega_m, \ \forall m \in \mathcal{M} \tag{3}$$

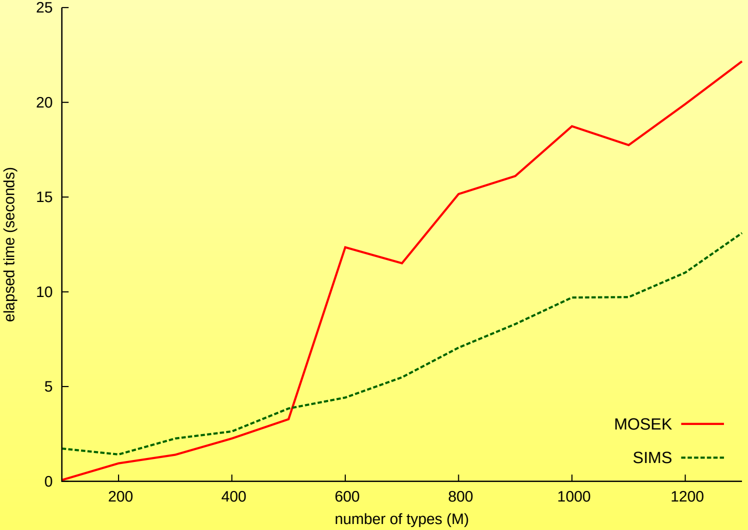
$$x_{im} \ge 0, \forall i \in \mathcal{N}, \ m \in \mathcal{M} \tag{4}$$

Optimality

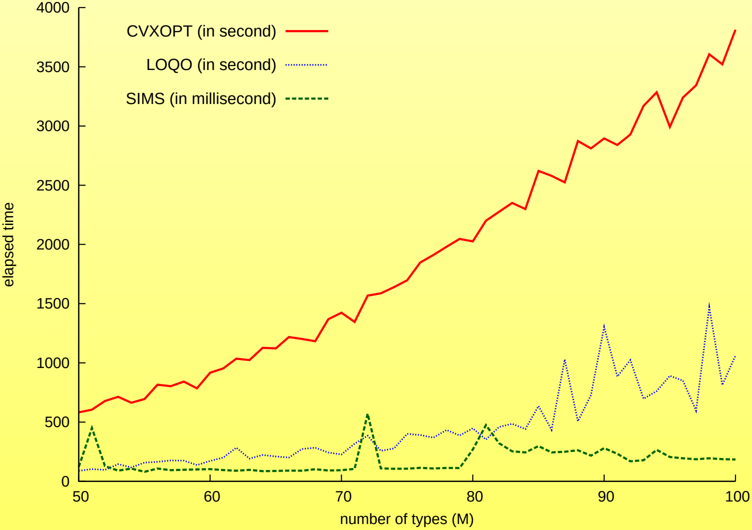
An allocation x is optimal if and only if x satisfies (3) and (4), and there exists a shadow price vector $(\lambda_1, \lambda_2, ..., \lambda_M)$ such that

$$\begin{cases} x_{im} > 0 \Rightarrow \frac{\partial Q_i(x_i)}{\partial x_{im}} = \lambda_m \\ x_{im} = 0 \Rightarrow \frac{\partial Q_i(x_i)}{\partial x_{im}} \le \lambda_m \end{cases}, \ \forall i, m \tag{5}$$

How about Convex Optimization?



How about Convex Optimization?



Example 1

Participants' Profiles and Supply

$$Q_i(x) = V_i\left(1 - e^{-\sum_{m=1}^{4}\alpha_{im}x_{im}}\right)$$

$$\mathbf{V} = \begin{pmatrix} 2 \\ 1 \\ 1.5 \\ 1.2 \end{pmatrix},\ \alpha = \begin{bmatrix} 0.30 & 0.16 & 0.10 & 0.20 \\ 0.20 & 0.50 & 0.12 & 0.05 \\ 0.13 & 0.10 & 0.40 & 0.08 \\ 0.06 & 0.10 & 0.20 & 0.30 \end{bmatrix},\ w = \begin{pmatrix} 12 \\ 8 \\ 6 \\ 6 \end{pmatrix}.$$

Pareto Optimality

$$x^0 = \begin{bmatrix} 5.4368 & 5.0289 & 1.3382 & 3.8107 \\ 2.6622 & 2.5018 & 0 & 0 \\ 3.9009 & 0.4693 & 3.0811 & 0 \\ 0 & 0 & 1.5807 & 2.1893 \end{bmatrix}, \ S = 4.7556.$$

- What if participant 1 gives one unit of type 2 impression to participant 2 in return for one unit of type 1 impression?
- Both will be better off and the social welfare will increase to 4.8085.

$$x^* = \begin{bmatrix} 11.823 & 0 & 0 & 0 \\ 0 & 6.7119 & 0 & 0 \\ 0.177 & 0 & 6 & 0 \\ 0 & 1.2881 & 0 & 6 \end{bmatrix}, \ S = 5.3001$$

Outline

Indicator Matrix

An *indicator matrix* $\mathbf{I}$ is an $N \times M$ matrix with binary elements $\delta_{im} \in \{0, 1\}$. An allocation $\mathbf{x}$ is *restricted* by $\mathbf{I}$, denoted as $\mathbf{x} \in_r \mathbf{I}$, if

$$\delta_{im} = 0 \Rightarrow x_{im} = 0, \forall i, m$$

Example 2

$$\mathbf{I}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Indicator Matrix and RNAS

Given an indicator matrix $\mathbf{I}$, we can define a *restricted* NAS problem:

$$
\begin{aligned}
\text{(RNAS)} \quad \max_{\{x_{im}\}} \sum_{i \in \mathcal{N}} Q_i \left(\sum_{m \in \mathcal{M}} \alpha_{im} x_{im} \right) \\
s.t. \sum_{i \in \mathcal{N}} x_{im} \le \omega_m, \ \forall m \in \mathcal{M} \\
x_{im} \ge 0, \forall i \in \mathcal{N}, \ m \in \mathcal{M} \\
\mathbf{x} \in_r \mathbf{I}
\end{aligned}
$$

Optimality and Pareto Optimality

Definition

- An indicator matrix $\mathbf{I}$ is said to be *optimal* if the solution to an NAS problem restricted by $\mathbf{I}$ also solves the original NAS problem.
- An indicator matrix $\mathbf{I}$ is said to be *Pareto optimal* if all allocations restricted by $\mathbf{I}$ are Pareto optimal.

Price Ratio Vector

A strictly positive vector $\mathbf{p} = (p_1, p_2, \cdots, p_M)$ is a price ratio vector for an indicator matrix $\mathbf{I}$ if for any $i \in \mathcal{N}$, $m, n \in \mathcal{M}$,

$$\frac{\alpha_{im}}{\alpha_{in}} \geq \frac{p_m}{p_n}, \text{whenever } \delta_{im} = 1 \tag{6}$$

- Intuitively, if price ratios between different types of goods are defined by $\mathbf{p}$, no agent would be interested in trading her currently allocated goods for any other type of goods.
- If $\mathbf{p}$ is a price ratio vector, so is $\lambda\mathbf{p}$ for any $\lambda > 0$. We say the price ratio vector for $\mathbf{I}$ is *unique* if all the price ratio vectors for $\mathbf{I}$ are proportional to each other.

Existence and Uniqueness of Price Ratio Vector

Example 3

Given the participants' profiles in Example 1 and $\mathbf{I}^*$, $\mathbf{I^1}$, $\mathbf{I^2}$ below,

$$\mathbf{I}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Any vector $\mathbf{r} = (13, r, 40, 3r)$ with $10 \leq r \leq 20$ is a price ratio vector for $\mathbf{I}^*$;
- There is no price ratio vector for $\mathbf{I^1}$ and $\mathbf{I^2}$.

Pareto Optimality and Price Ratio Vector

Theorem 1

An indicator matrix $\mathbf{I}$ is Pareto optimal if and only if there exists a price ratio vector for $\mathbf{I}$.

Comments

- Theorem 1 suggests any Pareto-optimal allocation supports a price ratio vector.
- Later we show that if a Pareto-optimal indicator matrix is optimal, then one of its price ratio vectors is proportional to the competitive equilibrium price vector.

Theorem 2

Let **I** be a Pareto-optimal indicator matrix and **p** be an associated price ratio vector. Define the supply ω_z and valuation functions $Q_i^z(\cdot), i \in \mathcal{N}$, for a "standardized" type as:

$$\omega_z \equiv \sum_{m \in \mathcal{M}} \omega_m p_m \tag{7}$$

$$Q_i^z(z_i) \equiv \begin{cases} Q_i\left(\frac{\alpha_{im}}{p_m} z_i\right), & \textit{if there exists some } m \textit{ such that } \delta_{im} = 1 \\ 0, & \textit{otherwise} \end{cases} \tag{8}$$

Theorem 2 (continued)

Let $\mathbf{z}^*$ be the solution to the following standardized single-type NAS problem

$$\max_{\{z_i\}} \quad \sum_{i\in\mathcal{N}} Q_i^z(z_i) \tag{9}$$
$$s.t. \quad \sum_{i\in\mathcal{N}} z_i \le \omega_z,\ z_i \ge 0, \forall i \in \mathcal{N}$$

and $\mathbf{x}$ be an allocation restricted by $\mathbf{I}$ that satisfies the following system of linear equations:

$$\begin{cases} \sum_{m\in\mathcal{M},\delta_{im}=1} p_m x_{im} = z_i^*, \forall i \in \mathcal{N} \\ \sum_{i\in\mathcal{N},\delta_{im}=1} x_{im} = \omega_m, \forall m \in \mathcal{M} \end{cases} . \tag{10}$$

The allocation $\mathbf{x}$ is a solution to the original NAS problem if it is non-negative.

Implications of Theorem 2

- Given a price ratio vector, we can convert a multi-type NAS problem to a standard single-type one, a huge reduction of dimension!
- In the standardized economy, the total supply is the sum of supplies of all types weighted by the price ratio vector. The system of linear equations allows us to recover an allocation $\mathbf{x}$ restricted by $\mathbf{l}$.
- If both $\mathbf{l}$ and the associated $\mathbf{p}$ are chosen "correctly", the allocation $\mathbf{x}$ recovered from the standard single-type NAS problem is a solution to the original NAS problem and $\mathbf{p}$ is proportional to the competitive equilibrium prices (or shadow prices for M types of goods).

Implications of Theorem 2

Five-step procedure

- Identify a Pareto-optimal indicator matrix $\mathbf{I}$;
- Use a price ratio vector for $\mathbf{I}$ to do standardization;
- Solve the standardized sing-type NAS problem;
- Solve the system of linear equations for the original allocation;
- Check the non-negativity of the solution.

Challenges

- How to obtain a price ratio vector from a PO indicator matrix
  - Solution: Regularity
- How to find the first (or the next) PO indicator matrix
  - Solution: SIMS algorithm

Connectivity

Definition

Typies m and n are *connected via agent* i, denoted as $m \overset{i}{\rightleftharpoons} n$, if agent i is permitted to hold both m and n under indicator matrix $\mathbf{I}$.

Connectivity Graph

Based on the connectivity information in $\mathbf{I}$, we can construct a undirected *connectivity graph* G in which

- each node represents a type, and
- An edge with label i exists between nodes m and n if $\delta_{im} = \delta_{in} = 1$.

Note that multiple edges with different labels can exist between a pair of nodes.

Connectivity Graph

$$\mathbf{I}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

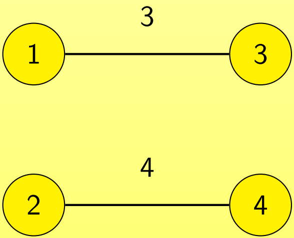


Figure 1 : The connectivity graph of $\mathbf{I}^*$

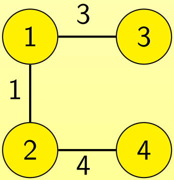
Connectivity

An indicator matrix $\mathbf{I}$ is *connected* if the connectivity graph generated by $\mathbf{I}$ is connected.

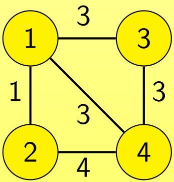
Which of the following indicator matrix is connected?

$$\mathbf{I}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^\mathbf{1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^\mathbf{2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Connectivity Graph



(a) The connectivity graph of $\mathbf{I}^1$



(b) The connectivity graph of $\mathbf{I}^2$

Example 4

Consider an example with two agents and two types of goods. Let

$$\mathcal{U}_1(\mathbf{x}_1) = Q_1(x_{11} + x_{12}),\ \mathcal{U}_2(\mathbf{x}_2) = Q_2(x_{21} + \beta x_{22}).$$

Consider five connected indicator matrices

$$\mathbf{I}^a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{I}^b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{I}^c = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\mathbf{I}^d = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{I}^e = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We can rule out $\mathbf{I}^a$ without sacrificing optimality.

Regularity

Definitions

- Given an indicator matrix $\mathbf{I}$, a type m has a *regular* connection with a connected component S ($m \notin S$) if
  - m is connected to at least one member of S,
  - all of m's connections to S are via the same agent.
- A connected indicator matrix $\mathbf{I}$ is *regular* if each type has a regular connection with each of the connected components among the remaining types.
- An indicator matrix $\mathbf{I}$ is *regular* if all of its connected components are regular.

Regularity

Which of the following indicator matrix is regular?

$$\mathbf{I}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^{\mathbf{1}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{I}^{\mathbf{2}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Proposition

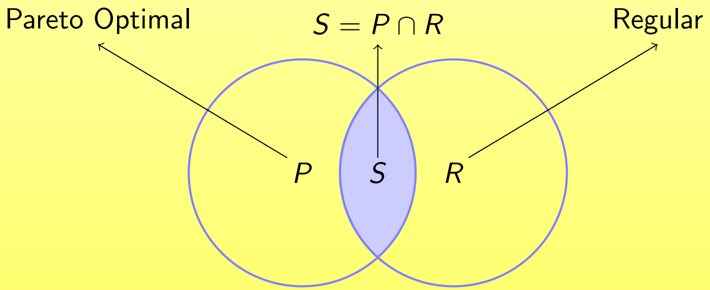
Let $\mathbf{I}$ be a connected and regular indicator matrix. Suppose there are L connections in $\mathbf{I}$: $m_1 \overset{i_1}{\leftrightarrow} n_1,\ m_2 \overset{i_2}{\leftrightarrow} n_2, ...,\ m_L \overset{i_L}{\leftrightarrow} n_L$. Then there exists a vector $\mathbf{p} = (p_1, p_2, \cdots, p_M)$, called a *pseudo price ratio vector*, that satisfies the following L equations:

$$\frac{p_{m_l}}{p_{n_l}} = \frac{\alpha_{i_l m_l}}{\alpha_{i_l n_l}}, \forall l = 1..L. \tag{11}$$

Such a pseudo price ratio vector is unique (in the same sense of uniqueness as a price ratio vector). Furthermore, if $\mathbf{I}$ is also Pareto optimal, then the pseudo price ratio vector is the unique price ratio vector for $\mathbf{I}$.

Theorem 3

If a Pareto-optimal allocation $\mathbf{x}$ is not regular, then there exists a regular Pareto-optimal allocation $\mathbf{x}'$ such that all agents are indifferent between $\mathbf{x}$ and $\mathbf{x}'$.



Implications of Theorem 3

- Since the optimal allocation must be Pareto optimal and each Pareto-optimal allocation must have an equivalent regular allocation, it is sufficient to search among regular indicator matrices.
- Theorem 3 plays a similar role in solving NAS as the fundamental theorem of linear programming does in solving linear programming problems.

Proposition

If the indicator matrix $\mathbf{I}$ is regular, then there exists a unique solution to the system of linear equations defined by (10), where $\mathbf{p}$ is a pseudo price ratio vector for $\mathbf{I}$.

Outline

Decomposition

Suppose the connectivity graph has J $(1 \leq J \leq M)$ connected components.

- We let $\mathcal{M}_j$ be the nodes in the jth component and $\mathcal{N}_j$ be the set of *affiliated* agents – that is, agents who may hold at least one type in $\mathcal{M}_j$.
- By construction, each agent is affiliated with at most one component.
- Let $\mathbf{I}_j$ denote a submatrix of $\mathbf{I}$ that consists of rows $\mathcal{N}_j$ and columns $\mathcal{M}_j$.
- In this way, we decompose the original problem into J subproblems.
- In the j-th sub problem, we allocate typies $\mathcal{M}_j$ among agents $\mathcal{N}_j$, subject to indicator matrix $\mathbf{I}_j$ that generates a connected connectivity graph.

Example 4

We illustrate the steps of our standardization technique using the profile in Example 1 and the Pareto optimal indicator matrix $\mathbf{I}^*$ below,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{V} = \begin{pmatrix} 2 \\ 1 \\ 1.5 \\ 1.2 \end{pmatrix}, \ \alpha = \begin{bmatrix} 0.30 & 0.16 & 0.10 & 0.20 \\ 0.20 & 0.50 & 0.12 & 0.05 \\ 0.13 & 0.10 & 0.40 & 0.08 \\ 0.06 & 0.10 & 0.20 & 0.30 \end{bmatrix}, \ w = \begin{pmatrix} 12 \\ 8 \\ 6 \\ 6 \end{pmatrix}.$$

Step 1: Decomposition

The first step is to decompose $\mathbf{I}$ into two submatrice.

$\mathbf{I}_1$:		Type 1	Type 3
	Participant 1	1	0
	Participant 3	1	1

$\mathbf{I}_2$:		Type 2	Type 4
	Participant 2	1	0
	Participant 4	1	1

Standardization

Subproblem $\mathbf{I}_2$

- Based on α_{42} and α_{44}, we have $r_2 = 1$, $r_4 = 3$
- The standardized supply of impressions is $\acute{w} = 8r_2 + 6r_4 = 26$.
- Participant 2 and 4's utility functions for standardized impressions are

$$\acute{u}_2(\acute{x}_2) = 1 - e^{-0.5\acute{x}_2}, \quad \acute{u}_4(\acute{x}_4) = 1.2\left(1 - e^{-0.1\acute{x}_4}\right).$$

- The optimal allocation for this homogeneous problem is

$$\acute{x}_2^* = 6.7119,\ \acute{x}_4^* = 19.2881.$$

Standardization

Recovering the Original Allocation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{22} \\ x_{42} \\ x_{44} \end{pmatrix} = \begin{pmatrix} 6.7119 \\ 19.2881 \\ 8 \end{pmatrix}$$

where

- the first two equations are feasibility condition for each participant regarding the conversion between the original allocation and the allocation for standardized impressions.
- The third equation is the feasibility condition for type 2 impression.

The solution to the above system of linear equations is:

$$\begin{pmatrix} x_{22} & x_{24} \\ x_{42} & x_{44} \end{pmatrix} = \begin{pmatrix} 6.7119 & 0 \\ 1.2881 & 6 \end{pmatrix}.$$

Combining Solutions of the Subproblems

Subproblem $\mathbf{I}_1$

With similar procedure we have

$$\begin{pmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{pmatrix} = \begin{pmatrix} 11.823 & 0 \\ 0.177 & 6 \end{pmatrix}.$$

Optimal Allocation

$$x^* = \begin{bmatrix} 11.823 & 0 & 0 & 0 \\ 0 & 6.7119 & 0 & 0 \\ 0.177 & 0 & 6 & 0 \\ 0 & 1.2881 & 0 & 6 \end{bmatrix}, \; S = 5.3001$$

How to Search for the Optimal $\mathbf{I}^*$

- If $\delta_{im} = 1$ and $x_{im} < 0$, we adjust $\mathbf{I}$ by setting $\delta_{im} = 0$;
- If $\delta_{im} = 0$ and $\frac{\partial Q_i}{\partial x_{im}} > \lambda_m$, we adjust $\mathbf{I}$ by setting $\delta_{im} = 1$.
- Adjust one element of $\mathbf{I}$ a time;
- Adjust the element of $\mathbf{I}$ that is most unbalanced first;
- Ensure the regularity of $\mathbf{I}$ after each adjustment.

Example 5 – Initialization

Set the initial indicator matrix $\mathbf{I}$ such that $\delta_{ij} = 1$ if and only if $V_i \alpha_{ij} \geq V_k \alpha_{kj}$, $\forall k \in \mathcal{N}$.

$$\mathbf{I}^0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, X^0 = \begin{bmatrix} 12 & 0 & 0 & 6 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$MU^0 = \begin{bmatrix} 0.0049378 & 0.0026335 & 0.0016459 & 0.0032919 \\ 0.0036631 & 0.0091578 & 0.0021979 & 0.00091578 \\ 0.017690 & 0.013608 & 0.054431 & 0.010886 \\ 0.072000 & 0.12000 & 0.24000 & 0.36000 \end{bmatrix}$$

Example 5 – Iteration 1

$$\mathbf{I}^1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, X^1 = \begin{bmatrix} 12 & 0 & 0 & -3.3893 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 9.3893 \end{bmatrix}$$

$$MU^1 = \begin{bmatrix} 0.032291 & 0.017222 & 0.010764 & 0.021527 \\ 0.0036631 & 0.0091578 & 0.0021979 & 0.00091578 \\ 0.017690 & 0.013608 & 0.054431 & 0.010886 \\ 0.0043055 & 0.0071758 & 0.014352 & 0.021527 \end{bmatrix}$$

Example 5 – Iteration 2

$$\mathbf{I}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, X^2 = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$MU^2 = \begin{bmatrix} 0.016394 & 0.0087436 & 0.0054647 & 0.010929 \\ 0.0036631 & 0.0091578 & 0.0021979 & 0.00091578 \\ 0.017690 & 0.013608 & 0.054431 & 0.010886 \\ 0.011902 & 0.019836 & 0.039672 & 0.059508 \end{bmatrix}$$

Example 5 – Iteration 3

$$\mathbf{I}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, X^3 = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 6.7119 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 1.2881 & 0 & 6 \end{bmatrix},$$

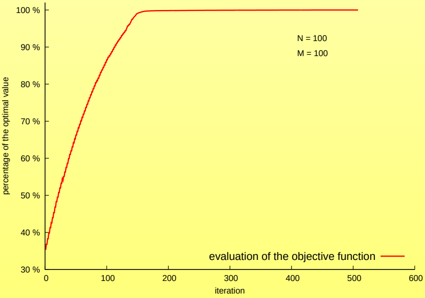
$$MU^3 = \begin{bmatrix} 0.016394 & 0.0087436 & 0.0054647 & 0.010929 \\ 0.0069754 & 0.017438 & 0.0041852 & 0.0017438 \\ 0.017690 & 0.013608 & 0.054431 & 0.010886 \\ 0.010463 & 0.017438 & 0.034877 & 0.052315 \end{bmatrix}$$

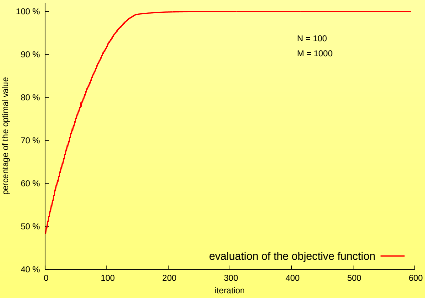
Example 5 – Iteration 4

$$\mathbf{I}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, X^4 = \begin{bmatrix} 11.823 & 0 & 0 & 0 \\ 0 & 6.7119 & 0 & 0 \\ 0.177 & 0 & 6 & 0 \\ 0 & 1.2881 & 0 & 6 \end{bmatrix},$$

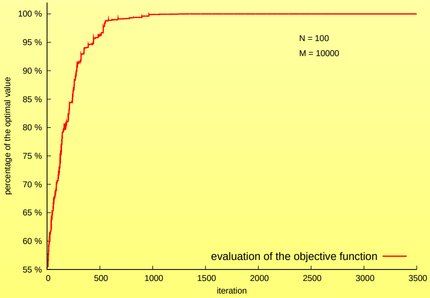
$$MU^4 = \begin{bmatrix} 0.017288 & 0.0092202 & 0.0057626 & 0.011525 \\ 0.0069754 & 0.017438 & 0.0041852 & 0.0017438 \\ 0.017288 & 0.013298 & 0.053193 & 0.010639 \\ 0.010463 & 0.017438 & 0.034877 & 0.052315 \end{bmatrix}.$$

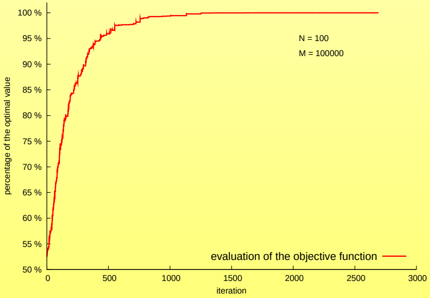
Convergence



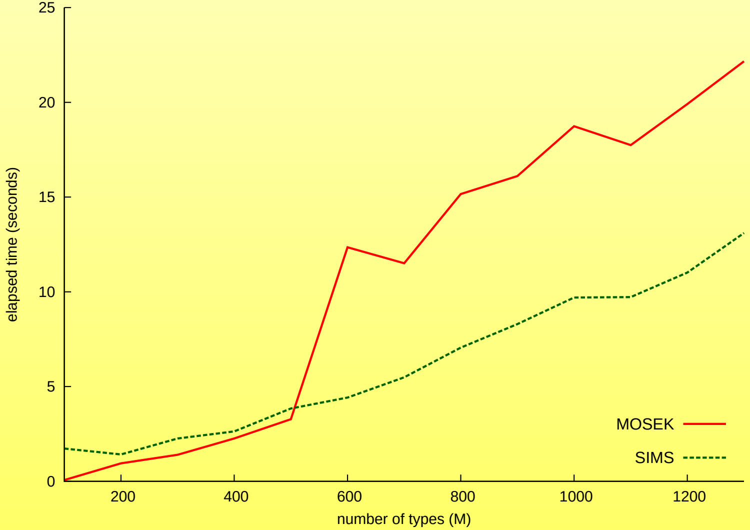


Convergence

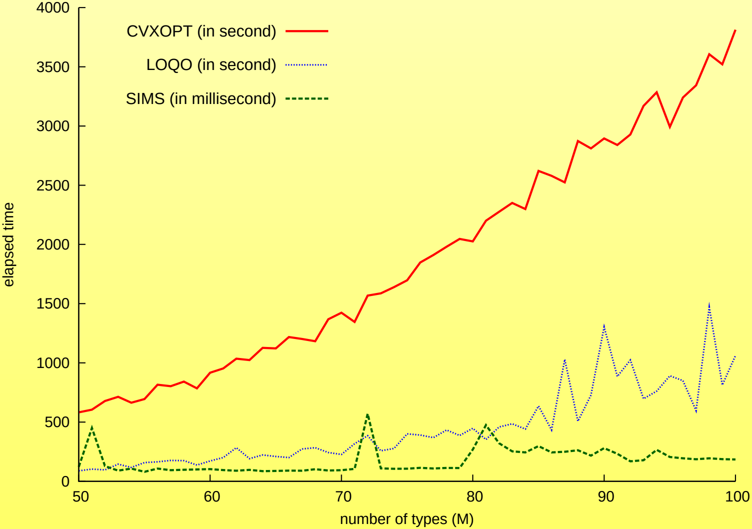




Comparison with MOSEK ($N = 100$)



Comparison with CVXOPT andLOQO ($N = 50$)



Thank you!