Optimal Securitization with Heterogeneous Investors

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Optimal Security Design

- Risk Sharing
  - Allen and Gale (1989), Whinton (1995)
- Liquidy and Asymmetric Information
  - Gorton and Pennachi (1990), Boot and Takor (1993), DeMarzo and Duffie (1999), Fulghieri and Lukin (2001), DeMarzo (2005), Axelson (2007)
- Moral Hazard
  - Hartman-Glaser, Piskorski, and Tchistyi (2009), Tchistyi (2009), and Piskorski and Tchistyi (2009)

The Model

- time periods $t = 0\,,\,1$
- issuer S has
  - cash flows X at time 1 with $\text{esssup} X \;=\; \bar{X}$
  - income stream $(w_0\,,\,w_1)$
  - utility

$$u_S(c_{0S}) \;+\; e^{-\rho_S}\,E[u_S(c_{1S})]$$

- N heterogeneous investors
- investor i has
  - endowment (w_{0i}, w_{1i})
  - utility

$$u_i(c_{0i}) \;+\; e^{-\rho_i}\,E[u_i(c_{1i})]$$

The Problem of Security Design. I

- the issuer creates a basket F_i , $i = 1, \cdots, N$ of limited-liability securities backed by the asset X
- no asymmetric information
- limited liability for investors: $F_i \ge 0$
- limited liability for the issuer

$$F = \sum_{i=1}^{N} F_i \le X.$$

- the issuer retains the residual cash flow $X - F$.

The Problem of Security Design. II

- the issuer is a monopolist
- he offers a security F_i to investor i, and the investor offers him the price $P_i \;=\; P_i(F_i)$
- variable cost of issuance: $C_i \;=\; \alpha\, P_i$ for some $\alpha \;\in\; (0,1)$
- investor i takes any contract satisfying satisfying the participation constraint

$$u_i(c_{0i}) \;+\; e^{-\rho_i}\, E[u_i(c_{1i})] \;\geq\; L_i,$$

- $$c_{0i} \;=\; w_0 \;-\; P_i \;,\; c_{1i} \;=\; w_{1i} + F_i(X)$$

  is the investor's consumption after entering the contract

- $$L_i \;=\; u_i(w_{0i}) \;+\; e^{-\rho_i}\, u_i(w_{1i})$$

  is the investor's reservation utility

The Problem of Security Design. III

- monopolistic price

$$P_i(F_i) \; = \; w_{0i} \; - \; v_i(L_i \, - \, e^{-\rho_i} \, E[u_i(w_{1i} + F_i(X))]) \, ,$$

where v_i is the inverse of the investor's utility,

$$v_i(u_i(x)) \; = \; x.$$

- given the contracts $(P_i \, , \, F_i) \, , \, i \, = \, 1 \, , \, \cdots \, , \, N$, the issuer's consumption is given by:

$$c_{0S} \; = \; w_0 \, + \, (1 \, - \, \alpha) \sum_{i=1}^{N} P_i, \qquad c_{1S} \; = \; w_1 \, + \, X \, - \, F(X)$$

- the issuer's securitization problem is to design the basket (F_i) so as to maximize his utility,

$$u_S(c_{0S}) \; + \; e^{-\rho_S} \, E\big[\, u_S(c_{1S}) \,\big]$$

Solution to the Single Investor Problem. I

- $g(a,x)$ is the unique solution to:

$$a\,u_B'(w_{1B} + g) - u_S'(w_1 + x - g) = 0$$

Theorem

(1) If

$$\rho_S - \rho_B > K_{\max},$$

then full selling is optimal,

$$F(X) = X\,;$$

(2) if

$$K_{\max} > \rho_S - \rho_B > K_{\mathrm{mid}},$$

then

$$F_a(X) = \begin{cases} X & , X \le Z(a) \\ g(a,X) & , X > Z(a) \end{cases}\;;$$

Solution to the Single Investor Problem. II

(3) if

$$K_{\text{mid}} > \rho_S - \rho_B > K_{\min},$$

then

$$F_a(X) = \begin{cases} 0 & , X \le Z(a) \\ g(a, X) & , X > Z(a) \end{cases} ; \quad \text{and}$$

(4) if

$$K_{\min} > \rho_S - \rho_B,$$

then there is no trade, that is, $F(X) = 0$.

Finding the threshold

$$Z(a) \;=\; \begin{cases} I_B(a^{-1}u_S'(w_1)) \,-\, w_{1B} & \text{in case (2)} \\ I_S(a\,u_B'(w_{1B})) \,-\, w_1 & \text{in case (3)} \end{cases},$$

and a is the unique solution to

$$a \;=\; \frac{(1-\alpha)\,e^{\rho_S}\,u_S'\,(w_0 \,+\, (1-\alpha)\,P_B(F_a(X)))}{e^{\rho_B}\,u_B'\big(w_{0B} \,-\, P_B(F_a(X))\big)},$$

where P_B is given by (6).

Marginal Rate of Intertemporal substitution

- $$\pi_B \, u_B'(c_{0B}) \; = \; e^{-\rho_B} \, u_B'(c_{1B}) \; \Leftrightarrow \; \pi_B \; = \; \frac{e^{-\rho_B} \, u_B'(c_{1B})}{u_B'(c_{0B})} \, .$$
- marginal trade happens if

$$\pi_B \; \geq \; \pi_S \, .$$

Security Slope

- slope

$$\frac{d}{dx} F_a(X) \;=\; \frac{R_B(c_{1B})}{R_B(c_{1B}) \,+\, R_S(c_{1S})},$$

- absolute risk tolerance

$$R_K(x) \;=\; -\frac{u'_K(x)}{u''_K(x)}\,,\; K \;=\; B\,,\, S.$$

- if the seller is risk-neutral then F is a standard debt:

$$F(X) \;=\; \min(X\,,\, d)$$

for some $d \,\geq\, 0$

Heterogeneous Investors

Proposition

- If all investors are risk neutral, then only the investor with the lowest discount rate will participate in a trade.
- If investors are risk averse and $\bar{X}$ is sufficiently large, then all investors will get a non-zero part of X.

Maximal Marginal Rates of Intertemporal Substitution (MMRIS)

- investors' MMRIS

$$Y_i \;=\; \frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(c_{0i})}$$

- issuer's MMRIS

$$Y_S \;=\; \frac{e^{-\rho_S}\, u_S'(w_1)}{(1-\alpha)\, u_S'(c_{0S})}\,.$$

- Lagrange multipliers

$$a_i \;\stackrel{def}{=}\; \frac{e^{\rho_S}\,(1-\alpha)\, u_S'(c_{0S})}{e^{\rho_i}\, u_i'(c_{0i})} \;=\; \frac{Y_i\, e^{\rho_S}\,(1-\alpha)\, u_S'(c_{0S})}{u_i'(w_{1i})}$$

Investors Ranking

For an investor i, we denote by $\text{rank}(i)$ the number that the investor will have when all investors are reordered so that Y_i are increasing in i.

J the number of investors for which Y_i is smaller than Y_S.

Solution to the Optimal Securitization Problem I.

There exist thresholds

$$0 \ = \ Z_{N+1} \leq Z_N \ \leq \ \cdots \ \leq \ Z_1 \leq Z_0 \ = \ \bar{X}$$

such that

(1) If $Y_i < Y_S$, then the investor i only participates in tranches $\mathrm{Tranche}_j = [Z_{j+1}, Z_j]$ with indices $j \ \leq \mathrm{rank}(i) \ - 1\,;$

(3) If $Y_i \geq Y_S$, then the investor i only participates in tranches $\mathrm{Tranche}_j$ with indices $j \ \leq \mathrm{rank}(i)\,;$

(4) The issuer fully sells the part of X below Z_{J+1} and retains a part of X for $X > Z_{J+1}$. That is,

$$F(X) \ = \ \sum_i F_i(X) \ = \ X$$

if $X \ \leq Z_{J+1}$ and $F(x) \ < \ x$ otherwise

Inside $\mathrm{Tranche}_j$:

- if $j \leq J$ then investors i with $rank(i) \geq j + 1$ and the issuer S share $\mathrm{Tranche}_j$ in a Pareto-efficient way
- if $j > J$ then investors i with $rank(i) \geq j$ share the (fully sold) $\mathrm{Tranche}_j$ in a Pareto-efficient way
- thus, optimal securities have a *subordinated structure*

Monotonicity

Proposition Optimal securities F_i and the retained part $X - F(X)$ are continuous and (weakly) monotone increasing in X;

Example.

Suppose that there are three investors with

$$Y_1 < Y_S < Y_2 < Y_3 \,.$$

Then, $J = 1$ and

$$\bar{X} \;=\; Z_0 \;>\; Z_1 \;>\; Z_2 \;>\; Z_3 \;>\; Z_4 = 0$$

if $\bar{X}$ is sufficiently large. In this case, optimal securities have the following structure:

- For $x \le Z_3$, $F_3(x) = x$, so investor 3 gets the whole super-senior tranche;
- For $x \in [Z_2, Z_3]$, $F_2, F_3 > 0$ and $F_2 + F_3 = X$, so investors 2 and 3 share the full pie;
- For $x \in [Z_1, Z_2]$, investors 2 and 3 still share the pie, but the issuer retains a part of it: $F_1 = 0$, $F_2, F_3 > 0$ and $F_2 + F_3 < X$; and
- Finally, for $x > Z_1$, $F_1, F_2, F_3 > 0$ and $F_1 + F_2 + F_3 < X$.

Securities Slopes

Proposition The slope $\frac{d}{dx}F_i(X)$ is given by

- For an investor i with $\mathrm{rank}(i) \geq J+1$,

$$\begin{cases} 0, & X \leq Z_{\mathrm{rank}(i)+1} \\ \frac{R_i(c_{1i})}{\sum_{j\,:\,\mathrm{rank}(j)\,\geq\,k} R_j(c_{1j})}, & X \in (Z_{k+1}, Z_k), J+1 \leq k \leq \mathrm{rank}(i) \\ \frac{R_i(c_{1i})}{R_S(c_{1S}) + \sum_{j\,:\,\mathrm{rank}(j)\,\geq\,k+1} R_j(c_{1j})}, & X \in (Z_{k+1}, Z_k), 0 \leq k \leq J \end{cases}$$

- For an investor i with $\mathrm{rank}(i) \leq J$,

$$\begin{cases} 0, & X \leq Z_{\mathrm{rank}(i)} \\ \frac{R_i(c_{1i})}{R_S(c_{1S}) + \sum_{j\,:\,\mathrm{rank}(j)\,\geq\,k+1} R_j(c_{1j})}, & X \in (Z_{k+1}, Z_k), 0 \leq k \leq \mathrm{rank}(i) - 1 \end{cases}$$

CARA Investors: Tranching Is Optimal

- A_i = risk aversion of investor i , A_S = issuer's risk aversion
- $I_{\text{rank}(i)\leq J}$ = indicator of investors with $\text{rank}(i) \leq J$

Proposition For each i, the investor i gets a *portfolio of tranches*

$$F_i = \sum_{k=0}^{\text{rank}(i)-I_{\text{rank}(i)\leq J}} \kappa_{ik} \, \text{Tranche}_k ,$$

with

$$\kappa_{ik} = \frac{A_i^{-1}}{A_S^{-1} \, I_{k\leq J} + \sum_{j:\text{rank}(j)\geq k+I_{k\leq J}}^{N} A_j^{-1}}$$

Finding the Thresholds

Theorem Optimal Tranche Thresholds can be calculated as a unique fixed point of an explicitly constructed contraction mapping.

An explicit iterative procedure for finding the thresholds.

Some Indicators

- $$Z_{\text{full selling}} \stackrel{def}{=} \max\{X : F(X) = X\}$$

- $$\#\{\text{senior}\} \stackrel{\text{def}}{=} \#\{\text{i} : \text{Y}_\text{i} > \text{Y}_\text{S}\}$$

  is the number of investors participating in the tranches that are fully sold
- if $Z_{\text{full selling}} = 0$, we define:

  $$Z_{\text{no trade}} = \max\{x : F(x) = 0\}$$

  to be the threshold Z_N of the super-senior tranche that is not sold at all
- for each investor i we define:

  $$\text{index}(i) = \begin{cases} 1 , & if \ \text{rank}(i) > J \\ 0 , & if \ \text{rank}(i) \leq J \end{cases} .$$

More Selling

We say that a change in the parameters of the model leads to more selling if it leads to an increase (in the weak sense) in:

- $\#\{\text{senior}\}$,
- $Z_{\text{full selling}}$,
- $\text{index}(i)$ for each i,

and to a decrease (in the weak sense) in $Z_{\text{no trade}}$.

Comparative Statics

- worse X quality $\Rightarrow$ More Selling
- higher discount rate ρ_S $\Rightarrow$ More Selling
- a decrease in w_0 $\Rightarrow$ More Selling
- an increase in the cost α $\Rightarrow$ More(Less) Selling if issuer's relative risk aversion is above(below) 1

Predictions

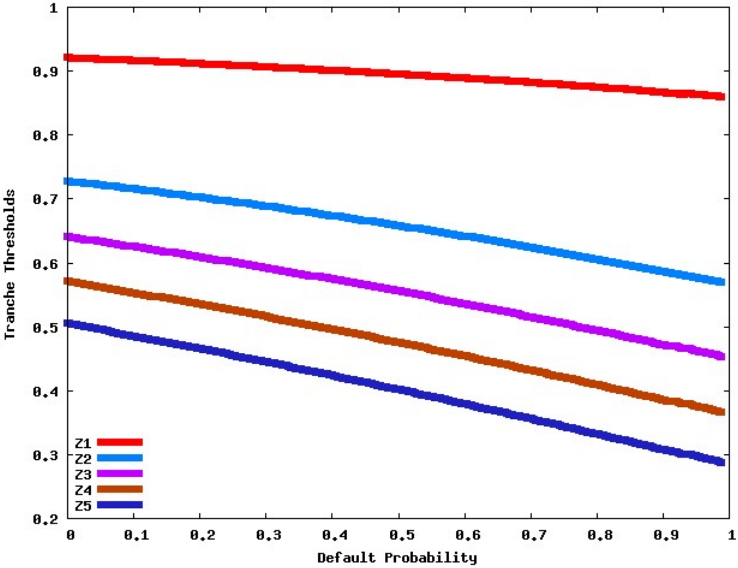
Proposition The following empirically observed properties hold true:

(1) Synthetic transactions are preferred to true-sale transactions for asset pools with high quality;

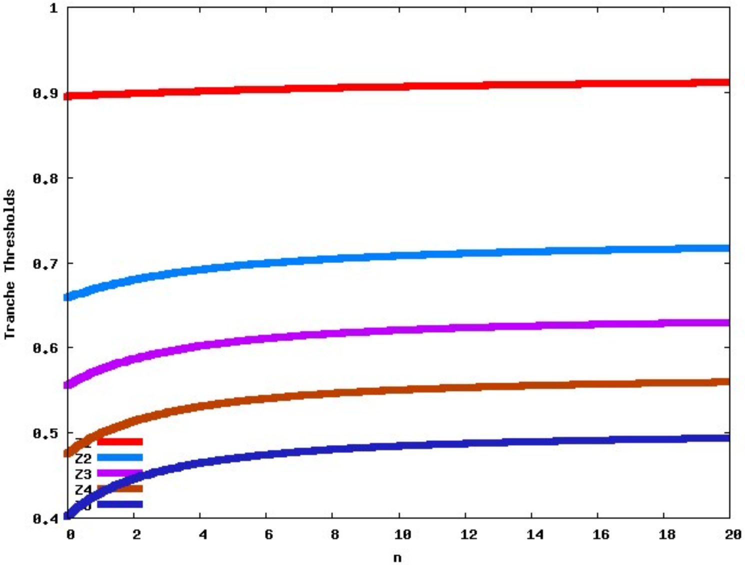
(2) In a synthetic transaction the size of the non-securitized super-senior tranche (TLP) increases with the quality of the asset pool; and

(3) Synthetic (true-sale) transactions are preferably used by banks with a strong (weak) rating.

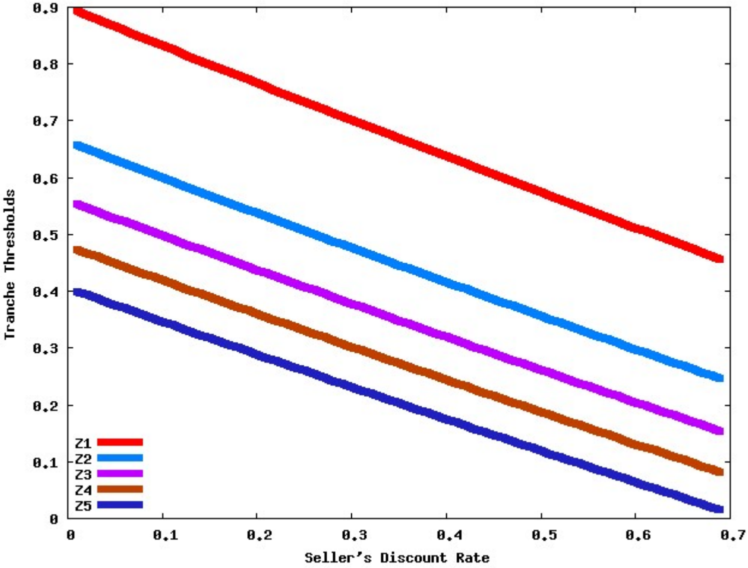
Change Distribution Through Default Probability



Change Distribution Through Skewness



The Effect of Issuer's Discount Rate

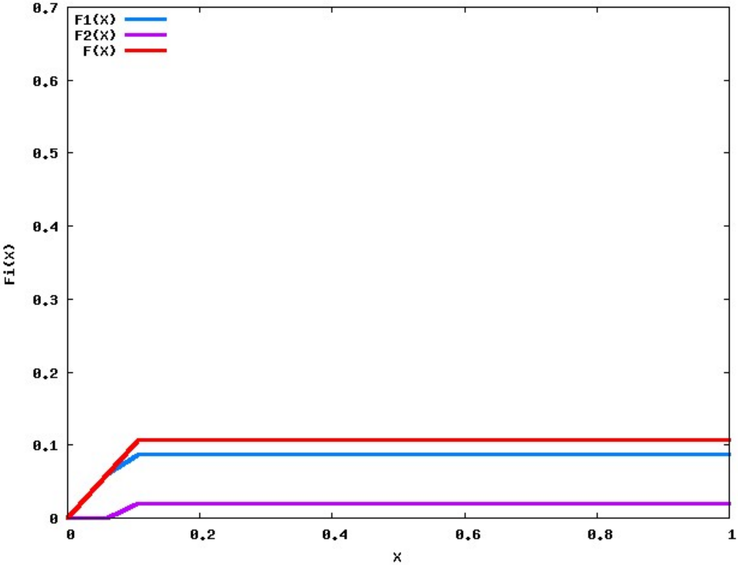


Risk Neutral Issuer

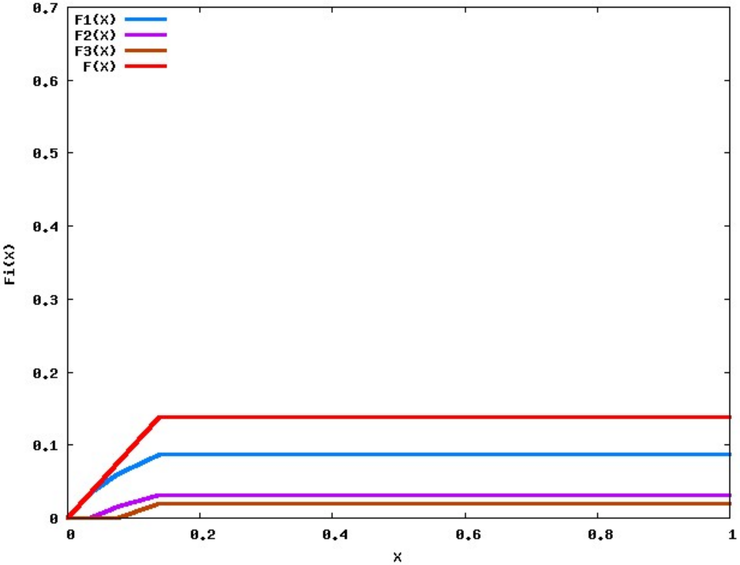
In this case

$$F(X) \; = \; \mathrm{Tranche}(0, Z_{J+1})$$

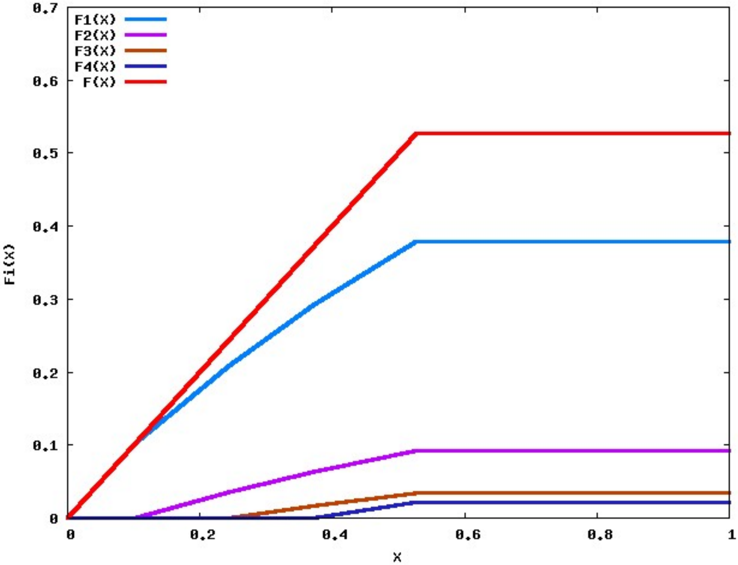
2 investors



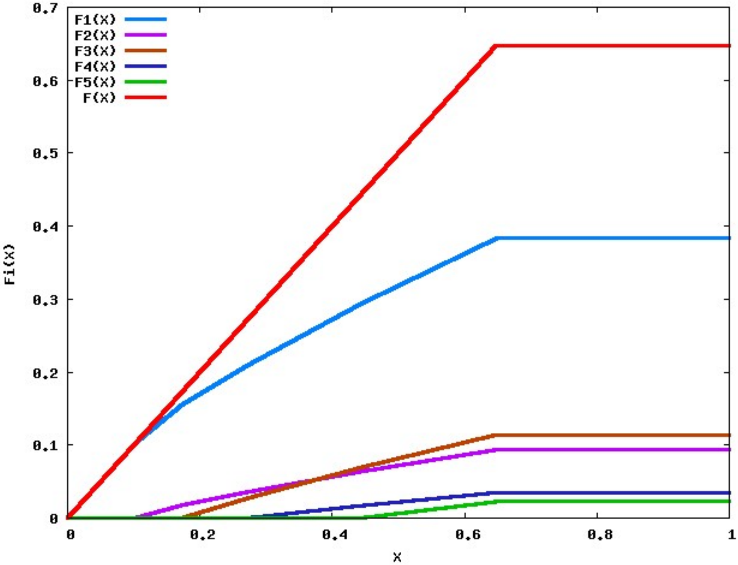
3 investors



4 investors



5 investors



Fixed Cost C of Issuing Securities I

Table: Agent Profiles with Risk-Averse Issuer

Agent	ρ	A	w_0	w_1
Issuer	0.08	1	0	0
Investor 1	0.01	0.3	0	0
Investor 2	0.04	0.4	0	0
Investor 3	0.06	0.6	0	0
Investor 4	0.08	0.1	0	0
Investor 5	0.1	0.1	0	0

Fixed Cost C of Issuing Securities II

Table: Optimal Selection of Investors with Fixed and Proportional Cost

C in %	0.01	0.1	0.5	2	13
Investors	1,2,3,4	1,2,4	1,4	4	–
Issuer's Expected Utility	0.599129	0.596688	0.589469	0.57357	0.49

Thank You!