# Optimal Auction Design for WiFi Procurement

Huaxia Rui

Simon Business School, The University of Rochester

Joint work with Liangfei Qiu and Andrew Whinston

# Agenda

1 Introduction

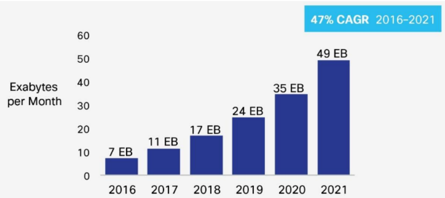
2 Model Setup

3 Global Auction

4 Glocal Auction

5 Conclusions

# Mobile Data Tsunami




# Challenge and Solutions

**Challenge**: Overloaded cellular networks, not enough bandwidth

## Solutions

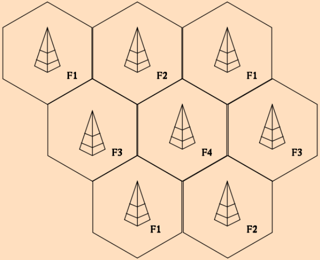
- **Spectrum**: acquire more spectrum — *more lanes?*
- **Infrastructure**: increase the number of cell towers — *interchange?*
- **Technology**: upgrade the network technology — *smaller car or driveless car?*
- **Mobile data offloading**: use of alternative network technologies for delivery of data *originally* targeted for cellular networks. — *road + subway ?*

# Mobile Data Offloading

- Range of a typial WiFi network: 120 ft to 300 ft
- Range of a typical cell tower: 1 – 2 miles

## An Old Idea

Mobile phone service had existed since the 1940s, but a small number of channels (e.g., 12 in NYC) were shared by customers.



# Homespots

Comcast's XFINITY and Cablevision's Optimum services have turned millions of home gateways into quasi-public hotspots.

Cablevision's Freewheel, includes unlimited data, talk and text for $9.95 a month for the company's broadband Internet subscribers and $29.95 for noncustomers.

# Third-party WiFi Hotspots

- Build your own hotspots
- Homespots
- Buy WiFi capacity from **third-party** hotspots.

Dong et al. (2014) and Iosifidis et al. (2013) proposed using auction for mobile data offloading.

# Research Question

## Why auction?

- Products with standardized characteristics
- The number of WiFi hotspots is large.

## Research Question

What is the **optimal** auction mechanism for a cellular service provider to procure WiFi capacity from third-party WiFi hotspots ?

# Challenge

A cellular tower can serve traffic in any region within its range (i.e., the cell sector), whereas WiFi hotspots can only serve local traffic.

## Naive Solutions

- Local auction: organize auctions based on WiFi region.
  - multiple auctions, one for each region.
  - **Disadvantage**: competition among hotspots will be reduced.
- Global auction: organize auction based on cellular sector.
  - one auction, with participation from hotspots from all regions.
  - **Disadvantage**: auction results may not be implementable.

## Our contribution

We propose **Glocal** auction, an optimal procurement mechanism that integrate local auctions and global auction.

# Agenda

# Cellular Service Provider

- A cellular network provides service to its customers who demand bandwidth.
- We think of the packets requested by the consumers as being serviced in a queuing system.
- $\mu$: the service capacity of the provider.
- $\lambda$: the demand arrival rate.

# Customers

## Expected Waiting Time

- $W(\mu)$: the expected waiting time a typical customer experiences.
- $W(\mu)$ should be decreasing in $\mu$ and be bounded below.

## Assumption 1

$$\frac{dW}{d\mu} < 0, \quad \frac{d^2W}{d\mu^2} > 0.$$

- Assumption 1 is satisfied for an $M/M/c$ queue.
- In Cheng et al. (2011), $W(\mu) = \frac{1}{\mu - \lambda}$.

# Congestion Cost

$\chi(W)$: the **indirect** cost borne by the **cellular service provider** due to customer waiting.

**Assumption 2**

$\chi(W)$ is strictly increasing and convex.

$$\frac{d\chi}{dW} > 0, \ \frac{d^2\chi}{dW} \geq 0.$$

- Cachon and Feldman (2011) assumed a linear structure.
- Define $\omega(\mu) \equiv \chi(W(\mu))$
- $\omega(\mu)$ is strictly decreasing and strictly convex in $\mu$.

# WiFi Hotspots

The cost of providing a capacity of $Q$ for WiFi hotspot $i$ is

$$C(Q, \theta_i) = \int_0^Q c(q, \theta_i) dq$$

where $\theta_i$ is hotspot $i$'s private information about its cost structure.

**Assumption 3: Hotspot Cost Structure**

- $c_q \geq 0$, $c_\theta \geq 0$, $c_{\theta\theta} \geq 0$, and $c_{q\theta} \geq 0$.
- $\theta_i$: independently and identically distributed with a distribution function $F(\cdot)$ defined on $[\underline{\theta}, \bar{\theta}]$.
- $H(\theta) \equiv F(\theta)/F'(\theta)$ is increasing in $\theta$.

- For any payment a hotspot receives for offloading mobile traffic, a proportion $(1 - \tilde{\alpha} \in (0, 1))$ of it goes to the hotspot's ISP.
- Define $\alpha = \tilde{\alpha}^{-1}$.

# WiFi Regions

- Literature suggests partitioning the service area of a cell tower (i.e., the cell sector) into several WiFi regions (e.g., Dong et al. 2014).
- Cellular capacity can serve traffic in any region, but WiFi hotspots in a region can only serve local traffic.

- $M$: number of WiFi regions in the cell sector.
- $y_m$: the total amount of WiFi capacity procured in region $m$.
- $N$: the total number of hotspots, in all regions.
- $E_m$: the set of hotspots in region $m$.
- $\omega_m(\cdot)$: region-specific congestion cost function.

# WiFi Regions

## The Goal

- Purchases WiFi capacity $(y_1, \cdots, y_M)$ for the $M$ regions from hotspots in these regions to supplement its cellular capacity.
- Minimize the total cost, including the total congestion cost $J(y_1, \cdots, y_M)$ and the procurement cost.

## Total Congestion Cost

$$J(y_1 \cdots, y_M) = \mathrm{Min}_{\mu_1, \cdots, \mu_M} \sum_{m=1}^{M} \omega_m(\mu_m + y_m)$$

$$s.t. \quad \sum_{m=1}^{M} \mu_m \leq \mu, \mu_m \geq 0, \text{for } m = 1, 2, ...M,$$

# Agenda

# Understanding Global Auction

Equivalent to relaxing the non-negativity constraints of $\mu_m$

$$J(y_1 \cdots, y_M) = \text{Min}_{\mu_1, \cdots, \mu_M} \sum_{m=1}^{M} \omega_m\big(\mu_m + y_m\big)$$

$$s.t. \quad \sum_{m=1}^{M} \mu_m \le \mu, \mu_m \ge 0, \text{for } m = 1, 2, ...M$$

## Definitions

- $y = \sum_{m=1}^{M} y_m$
- $\phi(\cdot)$ is the inverse of $\omega'(\cdot)$: $\phi(\omega'(x)) = x$
- $\Phi(\cdot) \equiv \sum_{m=1}^{M} \phi_m(\,\cdot\,)$
- $\Psi(\cdot)$ is the inverse of $\Phi(\cdot)$: $\Psi(\Phi(x)) = x$

# Proposition 1

## Optimal Cellular Resource Allocation

Under global auction, the optimal cellular resource allocation is given by

$$\mu_m^* = \phi_m\Big(\Psi(y+\mu)\Big) - y_m$$

The minimized congestion cost is

$$J(y) \equiv \sum_{m=1}^{M} \omega_m\Bigg(\phi_m\Big(\Psi(y+\mu)\Big)\Bigg).$$

Moreover, $J(y)$ is decreasing and convex.

# The Overall Procedure

- The cellular service provider announces a payment-bandwidth schedule $P_i = P(\theta_i, \theta_{-i})$, and a quantity schedule $q_i = Q(\theta_i, \theta_{-i})$;
- Hotspot $i$ truthfully reports the private cost parameter $\theta_i$ given $P(\theta_i, \theta_{-i})$ and $Q(\theta_i, \theta_{-i})$;
- Hotspot provides the WiFi capacity and receives the payment.
- Each WiFi region will be served by the procured WiFi capacity in the region and the cellular capacity of $\mu_m^*$.

We still need to design the mechanism.

# Payment Schedule

## Expected Profit of Reporting $\theta'$ for a Hotspot with $\theta$

$$\pi(\theta',\theta) \equiv \mathbb{E}_{-i}\big[\tilde{\alpha}P(\theta',\theta_{-i}) - C\big(Q(\theta',\theta_{-i}),\theta\big)\big],\ \pi(\theta) \equiv \pi(\theta,\theta)$$

## Necessary Condition for Incentive Compatibility (IC)

$$\begin{aligned}
\pi(\theta,\theta)-\pi(\theta,\theta') &\geq \pi(\theta,\theta)-\pi(\theta',\theta') \geq \pi(\theta',\theta)-\pi(\theta',\theta') \\
&\Rightarrow \mathbb{E}_{-i}[C\big(Q(\theta,\theta_{-i}),\theta'\big) - C\big(Q(\theta,\theta_{-i}),\theta\big)] \\
&\geq \pi(\theta,\theta)-\pi(\theta',\theta') \\
&\geq \mathbb{E}_{-i}[C\big(Q(\theta',\theta_{-i}),\theta'\big) - C\big(Q(\theta',\theta_{-i}),\theta\big)]. \\
\text{Send } \theta' \to \theta \Rightarrow \frac{d\pi(\theta)}{d\theta} &= -\mathbb{E}_{-i}\big[C_\theta\big(Q(\theta,\theta_{-i}),\theta\big)\big] \\
\Rightarrow \pi(\theta_i) &= \mathbb{E}_{-i}\Big[\int_{\theta_i}^{\theta^*} C_\theta\big(Q(\theta,\theta_{-i}),\theta\big)\,d\theta\Big].
\end{aligned}$$

# Quantity Schedule

Given $\pi(\theta)$, the condition for IC is

$$\mathbb{E}_{-i}\left[C(Q_i(x,\theta_{-i}),x)+\int_x^{\theta^*} C_\theta(Q_i(\theta,\theta_{-i}),\theta)d\theta - C(Q_i(x,\theta_{-i}),\theta_i)\right]$$
$$=\pi(x,\theta_i)\le\pi(\theta_i,\theta_i)=\mathbb{E}_{-i}\left[\int_{\theta_i}^{\theta^*} C_\theta(Q_i(\theta,\theta_{-i}),\theta)d\theta\right],$$

or equivalently
$$\mathbb{E}_{-i}\left[\int_{\theta_i}^{x} C_\theta(Q_i(x,\theta_{-i}),\theta)d\theta+\int_x^{\theta^*} C_\theta(Q_i(\theta,\theta_{-i}),\theta)d\theta\right]$$
$$\le \mathbb{E}_{-i}\left[\int_{\theta_i}^{x} C_\theta(Q_i(\theta,\theta_{-i}),\theta)d\theta+\int_x^{\theta^*} C_\theta(Q_i(\theta,\theta_{-i}),\theta)d\theta\right]$$

One sufficient condition for IC: $Q_i(\theta,\theta_{-i})$ is **decreasing** in $\theta$.

# Proposition 2

## Optimal Global Auction

The quantity schedule $q_i^* = Q^*(\theta_i, \theta_{-i})$ is determined by

$$-\Psi\left(\mu + \sum_{j=1}^{N} q_j^*\right) = \alpha c(q_i^*, \theta_i) + \alpha c_\theta(q_i^*, \theta_i) H(\theta_i), \forall i = 1, 2, \ldots N$$

The payment schedule $P_i = P^*(\theta_i, \theta_{-i})$ is given by:

$$P_i = \alpha\left(C(q_i^*, \theta_i) + \int_{\theta_i}^{\theta^*} C_\theta(Q^*(\theta, \theta_{-i}), \theta) d\theta\right), \forall i = 1, 2, \ldots N$$

The cellular service provider's expected gain:

$$J(0) - \mathbb{E}\left[J\left(\sum_{i=1}^{N} q_i^*\right) + \alpha \sum_{i=1}^{N} C(q_i^*, \theta_i) + \alpha \sum_{i=1}^{N} C_\theta(q_i^*, \theta_i) H(\theta_i)\right].$$

# A Closer Look at the Quantity Schedule

**$q_i^*$ is decreasing in $\theta_i$!**

$$-\Psi\left(\mu + \sum_{j=1}^{N} q_j^*\right) = \alpha c(q_i^*, \theta_i) + \alpha c_\theta(q_i^*, \theta_i) H(\theta_i), \forall i = 1, 2, \ldots N$$

Recall that $\Psi(q)$ is the inverse of $\Phi(\cdot) \equiv \sum_{m=1}^{M} \phi_m(\,\cdot\,)$ where $\phi_m(\cdot)$ is the inverse of $\omega_m'(\cdot)$.

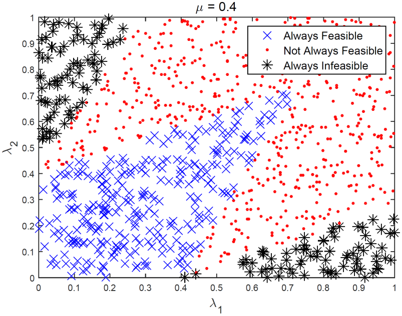
- $-\Psi(q)$: the marginal value of WiFi capacity when the total acquired WiFi capacity is $q$.
- The *virtual* marginal costs must all be equal to the marginal value of WiFi capacity.

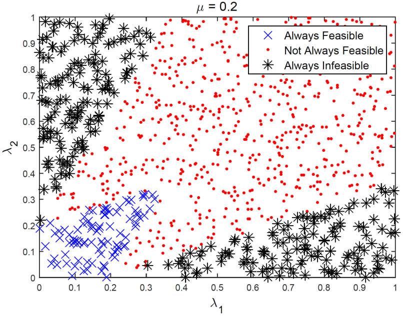
# Global Auction is Suboptimal!

We hoped that $\mu_m^* \geq 0$, or equivalently,

$$\mu \geq \Phi\Big(\omega_m'(y_m)\Big) - y, \ \forall m = 1, \cdots, M,$$

which we call the *feasibility condition*.



# Agenda

# A Way Out

- Given $(y_1, \cdots, y_M)$, let $\hat{\mu}_m$ be the optimal amount of cellular capacity allocated to region $m$ **ignoring** all the non-negativity constraint on $\mu_i$ (i.e., in the global auction)
- We already know $\hat{\mu}_m = \phi_m\Big(\Psi(y + \mu)\Big) - y_m$.

**Intuition**

- The optimal auction should be designed so that $\mu_m$ coincides with $\hat{\mu}_m$ whenever possible;
- Whenever $\hat{\mu}_m < 0$ under some realizations of $(\theta_1, \cdots, \theta_N)$, the optimal auction should adjust the quantity schedule to account for the corner solution in the second-stage optimization problem.

# The Optimal Auction Design Problem

$$
\begin{aligned}
\max_{\substack{q_i, i=1,\cdots,N \\ \mu_m, m=1,\cdots,M}} \Pi \quad &= \quad \mathbb{E}\Bigg[\sum_{m=1}^{M} -\omega_m\big(\mu_m + y_m\big) - \alpha \sum_{i=1}^{N} C(q_i, \theta_i) \\
& \qquad -\alpha \sum_{i=1}^{N} C_\theta(q_i, \theta_i) H(\theta_i)\Bigg], \\
s.t. \quad & \qquad \sum_{m=1}^{M} \mu_m \leq \mu, \\
& \qquad \mu_m \geq 0, \forall m = 1, 2, \cdots, M, \\
& \qquad y_m = \sum_{i \in E_m} q_i, \forall m = 1, 2, \cdots, M,
\end{aligned}
$$

# Optimality

- The variational calculus problem is degenerated and we can solve the problem through pointwise optimization over the space of $\Theta$.
- When $M = 2$, we can divide the space of $\Theta$ into two areas,

$$\begin{aligned}\Theta_1 &\equiv \{(\theta_1, \theta_2, \cdots, \theta_N) \big| \mu_1^* > 0, \mu_2^* > 0\},\\ \Theta_2 &\equiv \{(\theta_1, \theta_2, \cdots, \theta_N) \big| \mu_1^* = 0 \text{ or } \mu_2^* = 0\}.\end{aligned}$$

- If $\vec{\theta} \in \Theta_1$, the nonnegativity conditions of $\mu_m$ are not binding. We are back to global auction!
- If $\vec{\theta} \in \Theta_2$, then one of the nonnegativity constraint must be binding at the optimal. The coupling across region disappears and we are back to local auctions!

# Proposition 3

## Optimal Mechanism when $M = 2$

The quantity schedule $q_i^{**} = Q_i^{**}(\theta_i, \theta_{-i})$ is determined by

$$-\Psi\left(\mu + \sum_{j=1}^{N} q_j^{**}\right) = \alpha c(q_i^{**}, \theta_i) + \alpha c_\theta(q_i^{**}, \theta_i) H(\theta_i), \forall i = 1, \cdots, N$$

if $\hat{\mu}_1 \geq 0$ and $\hat{\mu}_2 \geq 0$, and is otherwise determined by

$$-\omega'_m\left(\mu \mathbf{1}_{\hat{\mu}_m > 0} + \sum_{j \in E_m} q_j^{**}\right) = \alpha c\left(q_i^{**}, \theta_i\right) + \alpha c_\theta(q_i^{**}, \theta_i) H(\theta_i), \forall i \in E_m$$

The optimal payment schedule $P_i^{**}(\theta_i, \theta_{-i})$, for $i = 1, 2, ...n$, is given by:

$$P_i^{**}(\theta_i, \theta_{-i}) = \alpha\left(C\left(q_i^{**}, \theta_i\right) + \int_{\theta_i}^{\theta^*} C_\theta\left(Q_i^{**}(\theta, \theta_{-i}), \theta\right) d\theta\right).$$

# Incentive Compatibility

The key to incentive compatibility is for $q_i^{**}(\theta_i)$ to be non-increasing.

## Local Monotonicity

- The optimal quantity schedule is a non-smooth function of $(\theta_1, \cdots, \theta_N)$.
- We already know, that within each segment, the optimal quantity schedule is decreasing in each $\theta_i$.

## Global Monotonicity?

- Is the optimal quantity schedule decreasing across segments?
- Will hotspots have incentive to lie to trigger a switch from global auction to local auctions or vice versa?

# Incentive Compatibility

## Continuity

- $q_i^{**}(\theta_i)$ is continuous everywhere.
- Continuity upgrades local monotonicity to global monotonicity.

- Whenever the feasibility condition is satisfied, the mechanism is equivalent to the global auction.
- Whenever the feasibility condition is violated, the mechanism is equivalent to allocating all cellular capacity to one region and to organize one *local* auction in each region.

# A Necessary Clarification

**There is only ONE actual auction!**

- The choice between a global auction and two local auctions is **endogeneously** determined by the auctioneer based on the realization of $(\theta_1, \cdots, \theta_N)$.
- From the perspective of a hotspot, ex ante, it does not know whether it will participate in a global auction or a local auction. It does not need to know.
- What matters to a hotspot is only the payment and quantity schedule announced by the auctioneer.

# What happens with $M > 2$? (Skip if out of time)

- The same idea of integrating local and global auction.
- The additional task is to optimally divide the set of regions into two subsets: one where a global auction will be held ($R_g$) and the other where local auctions will be held ($R_l$).

But the number of ways to divide the set of WiFi regions is

$$\frac{\left|2^{\{1,2,\cdots,M\}}\right|}{2} = \frac{2^M}{2} = 2^{M-1}$$

# Proposition 4

## Finding the largest feasible $R_g$

- Given $M \geq 2$ and $(\theta_1, \cdots, \theta_N)$, and suppose there are two different schemes of dividing the regions into global and local auctions.
- Both schemes lead to feasible allocation of cellular capacity: $(R_g, R_l)$ and $(\tilde{R}_g, \tilde{R}_l)$ where $\tilde{R}_g \subsetneq R_g$.
- The optimal gain corresponding to the auction design with $(R_g, R_l)$ is larger than the optimal gain corresponding to the auction design with $(\tilde{R}_g, \tilde{R}_l)$.

# From Combinatorial to Sequential

## Intuition

- Given Proposition 4, we clearly should start with $R_g = \{1, \cdots, M\}$.
- If this leads to infeasible allocation of cellular capacity, we have to shrink $R_g$.
- We probably should exclude those regions with $\mu_m^* < 0$ from $R_g$ to restore feasibility.

The main concern with the sequential procedure is whether "exclusion" should be irreversible.

# From Combinatorial to Sequential

Fortunately, the answer is yes.

- We can show that if a region is in $R_l$ at some stage, then it will be in $R_l$ in later stages had it remained in $R_g$.
- This guarantees the complexity is of the order of $M$.

## Defining the $k$-subproblem

- Let $q_{i,k}$ be determined by the following equation:

$$-\Psi\left(Y_k\right) = \alpha c(q_{i,k}, \theta_i) + \alpha c_\theta(q_{i,k}, \theta_i) H(\theta_i), \forall i \in \bigcup_{m \in R_g^k} E_m.$$

- $Y_k \equiv \sum_{m \in R_g^k} y_{m,k}$, $y_{m,k} = \sum_{i \in E_m} q_{i,k}$
- Let $\mu^*_{m,k} = \phi_m\big(\Psi(Y_k + \mu)\big) - y_{m,k}$
- Let $R_+^k \equiv \{m \in R_g^k | \mu_{m,k} \geq 0\}$, $R_-^k \equiv \{m \in R_g^k | \mu_{m,k} < 0\}$.

# Proposition 5

## Optimal Mechanism when $M \geq 2$

Given $M \geq 2$ and $(\theta_1, \cdots, \theta_N)$, the optimal quantity schedule $q_i^{**}$ is given by

$$
\begin{aligned}
-\Psi\Big(\mu + \sum_{j \in E_m, m \in R_g} q_j^{**}\Big) &= \alpha c(q_i^{**}, \theta_i) + \alpha c_\theta(q_i^{**}, \theta_i) H(\theta_i), \\
&\quad \forall i \in E_m, m \in R_g; \\
-\omega_m'\big(\sum_{j \in E_m} q_j^{**}\big) &= \alpha c\,(q_i^{**}, \theta_i) + \alpha c_\theta(q_i^{**}, \theta_i) H(\theta_i) \\
&\quad \forall i \in E_m, m \in R_l
\end{aligned}
$$

# Proposition 5 (continued)

**Optimal Mechanism when $M \geq 2$**

$R_g$ and $R_l$ is constructed through the following iterative procedure:

- (0): Let $k = M$, $R_g^M = \{1, 2, \cdots, M\}$, and $R_l^M = \emptyset$.
- (1): If $R_-^k = \emptyset$, let $R_g = R_g^k$ and $R_l = R_l^k$. Stop the procedure.
- (2): If $R_-^k \neq \emptyset$, let $R_g^{k-1} = R_+^k$ and $R_l^{k-1} = R_l^k \cup R_-^k$. Decrease $k$ by 1 and repeat (1).

The optimal payment schedule $P_i^{**}$, for $i = 1, 2, ...n$ is given by:

$$P_i^{**} = \alpha\left( C\left(q_i^{**}, \theta_i\right) + \int_{\theta_i}^{\theta^*} C_\theta\left(q_i^{**}, \theta\right) d\theta \right).$$

# Agenda

# Contributions

- Introduced a procurement auction framework for mobile data offloading.
- Solved the optimal auction design problem in a very general setting.
- Theoretical insights on the integration of global auction and local auction may apply to other auction design problems.

# Limitations

- We only considered one cellular service provider.
- We assumed the marginal cost function of all hotspots can be approximated using a one-parameter function family.
- We focused on the supply side by abstracting away the consumer side.

# Thank you!